Math 418: Problem Set 8.

Due date: In class on Wednesday, April 14.
Webpage: http://dunfield.info/418
Office hours: Monday 10-11, Tuesday 3-5.

All problems are from Dummit and Foote, *Abstract Algebra*, 3rd edition.

1. Fix a prime p. Show that the following subgroup of $\mathrm{GL}_2\mathbb{F}_p$ is solvable:

$$B = \left\{ \begin{pmatrix} x & z \\ 0 & y \end{pmatrix} \;\middle|\; x, y \in \mathbb{F}_p^\times, z \in \mathbb{F}_p \right\}$$

   Here, the group operation is just matrix multiplication.

2. Let G be a group. The *commutator* of two elements $g, h \in G$ is $ghg^{-1}h^{-1}$ and is denoted $[g, h]$. Let $[G, G]$ be the subgroup of G generated by all such commutators.

   (a) Prove that $[G, G]$ is normal in G and the quotient $G/[G, G]$ is abelian.

   Now consider the sequence of subgroups G^i where $G^0 = G$ and $G^1 = [G, G]$ and generally $G^{i+1} = [G^i, G^i]$. By part (a), we have

$$G = G^0 \triangleright G^1 \triangleright \cdots \triangleright G^i \triangleright \cdots$$

   (b) Suppose that some $G^i = \{1\}$. Prove that G is solvable. (In fact, the converse is true as well.)

   (c) Use (b) to prove that S_4 is solvable.

3. Prove that A_5 is simple using the following outline. Suppose $G \triangleleft A_5$ is normal subgroup which is not $\{1\}$.

   (a) Show that G contains some 3-cycle.

   (b) Show that G contains *every* 3-cycle.

   (c) Show that A_5 is generated by all 3-cycles.

   (d) Show that A_5 is simple.

   Alternatively, give a geometric proof using the fact that A_5 is the group of Euclidean isometries of a regular dodecahedron.

4. Section 14.7, #12. You'll need to use Theorem 18 from Section 4.5 for this to show that if p is a prime dividing the order of G, then G has an element of order p.

5. Section 14.7, #13.