Math 418: Takehome Midterm 1

Due date: In class on Wednesday, February 17.

Disclaimer, Terms, and Conditions: You may not discuss the exam with anyone except myself. You may *only* consult the following:

- The beloved(?) text, Dummit and Foote's *Abstract Algebra*.
- Your class notes and returned HW sets.
- My online class notes and HW solutions.

You can use any result in Chapters 7–9 and Sections 13.1–13.2 of Dummit and Foote, even if I didn't cover it in class. You can also use the result of any HW problem that was assigned, whether or not you did it.

Office hours: While discussion of these problems will be limited to clarification of their statements, I will also be happy to answer broader questions about the course material during my usual office hours (M 10-11, Tu 3-5, and by appointment).

1. Let F be a field. Consider the ring $R = F[[t]]$ of *formal power series* in t, namely things of the form
$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 t + a_2 t^2 + \cdots \quad \text{where } a_n \in F.$$
Here "formal" means the above "sum" is really just an infinite list of elements of F; there's no notion of convergence involved. Elements of R are added term by term, and multiplication is as if they were polynomials. More precisely
$$\sum_{n=0}^{\infty} a_n t^n \times \sum_{n=0}^{\infty} b_n t^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) t^n$$
It is clear that R is a commutative ring with unit.

   (a) Prove that α in R is a unit if and only if the constant term $a_0 \neq 0$. (Example: The inverse of $1 - t$ is $1 + t + t^2 + t^3 + t^4 + \cdots$.)

   (b) Prove that R is a Euclidean domain with respect to the norm $N(\alpha) = n$ if a_n is the first term of α that is non-zero. (If $F = \mathbb{C}$ and the power series converges near $t = 0$, then this norm is just the order of zero of the corresponding function at 0.)

   (c) In the polynomial ring $R[x]$, prove that $x^n - t$ is irreducible.

2. Let $R = \mathbb{Z}[i]$.

   (a) Prove that $R/(1+i)$ is a field of order 2.

   (b) Let $\pi \in R$ be irreducible. Consider the ideals $I_n = (\pi^n)$. Prove that $R/(\pi) \cong I_n / I_{n+1}$ as additive abelian groups. Hint: the isomorphism is multiplication by π^n.

   (c) Again for irreducible π, prove that $|R/(\pi^n)| = |R/(\pi)|^n$. Here $|\cdot|$ denotes the number of elements in a finite set. (This is a key step in proving that for *any* $\pi \in R$ that $|R/(\pi)| = N(\pi) = |\pi|^2$.)

   (d) For $\pi = 1 + i$, are $R/(\pi^3)$ and $\mathbb{Z}/8\mathbb{Z}$ isomorphic as rings?

3. Section 13.2, #8.
4. Section 13.2, #13.
5. Section 13.2, #20.