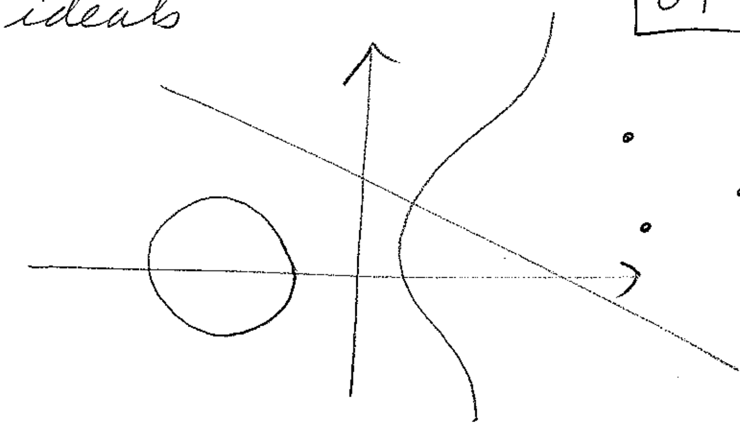
Lecture 32: Varieties and ideals



k - field

Affine Space: k^n

$I \subseteq k[x_1, \ldots, x_n]$

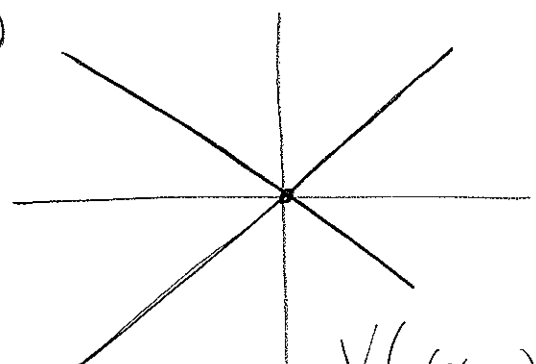
Algebraic Variety: $\mathbb{V}(I) = \{a \in k^n \mid f(a) = 0 \ \forall f \in I\}$

I might as well be an ideal.

Basic Props: ① $I \subseteq J \Rightarrow \mathbb{V}(I) \supseteq \mathbb{V}(J)$

② $\mathbb{V}(I) \cap \mathbb{V}(J) = \mathbb{V}(I \cup J) = \mathbb{V}(I + J)$

③ $\mathbb{V}(I) \cup \mathbb{V}(J) = \mathbb{V}(IJ)$



$V((x-y)(x+y))$

Let V be an alg. variety. Set

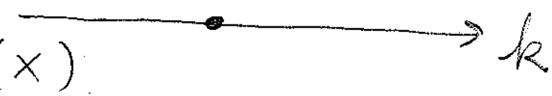
$\mathbb{I}(V) = \{f \in k[x_1, \ldots, x_n] \mid f(a) = 0 \text{ for all } a \in V\}$

Clearly: $\mathbb{I}(\mathbb{V}(I)) \supseteq I$.

Unfortunately, they're not always equal:

<u>Ex</u>: Consider $I = (x^2) \subseteq k[x]$.

$\mathbb{V}(I) = \{0\}$ and $\mathbb{I}(\mathbb{V}(I)) = (x)$



[In practice, this is a real problem...]

<u>Def</u>: Let I be an ideal in a comm. ring R.
Its radical is: $\operatorname{rad}(I) = \{a \in R \mid a^n \in I\}$

<u>Ex</u>: $\operatorname{rad}((x^2)) = (x)$

"Zero locus thm."

<u>Hilbert's Nullstellensatz</u>: Suppose k is alg. closed.
Then $\mathbb{I}(\mathbb{V}(I)) = \operatorname{rad}(I)$ for all ideals
$I \subseteq k[x_1, \ldots, x_n]$. Moreover

$$\left\{\begin{array}{c}\text{Algebraic varieties}\\ \text{in } k^n\end{array}\right\} \underset{\mathbb{V}}{\overset{\mathbb{I}}{\rightleftarrows}} \left\{\begin{array}{c}\text{radical ideals}\\ \text{in } k[x_1, \ldots, x_n]\end{array}\right\}$$

are inverse bijections.

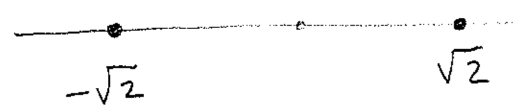
Note: $\mathbb{I}(\mathbb{V}(I)) \supseteq \operatorname{rad}(I)$

Pf: Suppose $f \in \operatorname{rad}(I)$ with $f^n \in I$.
If $a \in V(I)$, then $0 = f^n(a) = (f(a))^n$
$\Rightarrow f(a) = 0$. So $f \in \mathbb{I}(\mathbb{V}(I))$. ▨

For a proof of the Nullstellensatz, see DF, §15.3.

Ex: $I = (x^2 - 2) \subseteq \mathbb{Q}[x]$. Then
$\mathbb{I}(\mathbb{V}(I)) = \mathbb{Q}[x]$ since $V(I) = \emptyset$.

Ex: $I = (x^2 + 1) \subseteq \mathbb{R}[x]$



$\mathbb{I}(\mathbb{V}(I)) = \mathbb{R}[x]$

Nullstellensatz II: $k \subseteq \bar{k}$ with $\bar{k}$ algebraically closed. If $I \subseteq k[x_1, \ldots, x_n]$, then

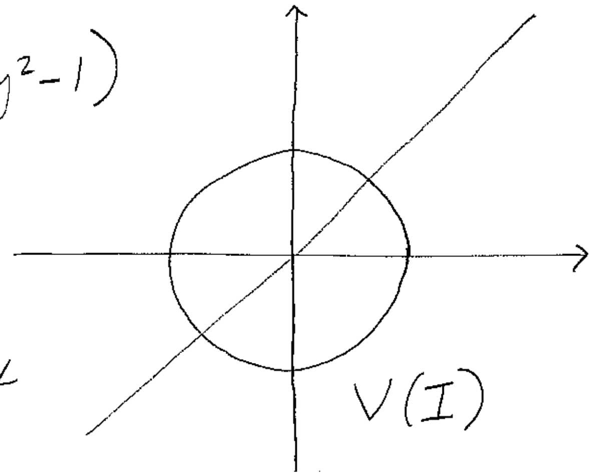
$$\mathbb{I}_k(\underbrace{\mathbb{V}_{\bar{k}}(I)}_{\subseteq \bar{k}^n}) = \operatorname{rad}(I)$$

Decomposing varieties:

$$I = (x^3 + xy^2 - yx^2 - y^3 - x + y) \subseteq \mathbb{R}[x,y]$$

See $\mathbb{V}(I) = \mathbb{V}(x-y) \cup \mathbb{V}(x^2+y^2-1)$

and in fact $I = (x-y)(x^2+y^2-1)$



What about $\mathbb{V}(x-y)$? Can it be written as a union of two varieties?

Def: A variety V is <u>irreducible</u> if whenever $V = V_1 \cup V_2$ for varieties V_i, then $V = V_1$ or $V = V_2$.

<u>Thm</u>: V is irreducible iff $\mathbb{I}(V)$ is prime.

<u>Proof</u>: ($\Longrightarrow$) Suppose $f_1 \cdot f_2 \in \mathbb{I}(V)$.

Set $V_i = V \cap \mathbb{V}(f_i) = \mathbb{V}((f_i) + \mathbb{I}(V))$

$= \{\text{pts of } V \text{ where } f_i = 0.\}$

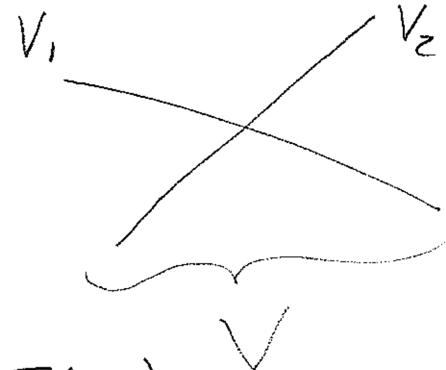
Now if $a \in V$, have $(f_1 f_2)(a) = f_1(a) f_2(a) = 0$

$\Rightarrow f_1(a) = 0$ <u>or</u> $f_2(a) = 0$. So

$$V = V_1 \cup V_2.$$

As V is irred have, say, $V = V_1$, i.e. $f_1(a) = 0$ for all $a \in V \Rightarrow f_1 \in \mathbb{I}(V)$. So $\mathbb{I}(V)$ is prime.

($\Leftarrow$) Suppose $V = V_1 \cup V_2$. Assume $V \neq V_1$.



As $V_1 \subsetneq V$, have $\mathbb{I}(V) \subsetneq \mathbb{I}(V_1)$, and so let $f_1 \in \mathbb{I}(V_1) \setminus \mathbb{I}(V)$. Let $f_2 \in \mathbb{I}(V_2)$. Then $f_1 f_2 = 0$ on $V \Rightarrow f_1 f_2 \in \mathbb{I}(V)$. As $\mathbb{I}(V)$ is prime, must have one $f_i \in \mathbb{I}(V)$, which must be f_2. Hence $\mathbb{I}(V_2) \subseteq I(V)$ and so $V_2 \supseteq V \Rightarrow V = V_2$.

$\blacksquare$