Lecture 33:

$I \subseteq k[x_1, \ldots, x_n]$

$\mathbb{V}(I) = \{a \in k^n \mid f(a) = 0 \ \forall f \in I\}$ – alg. variety

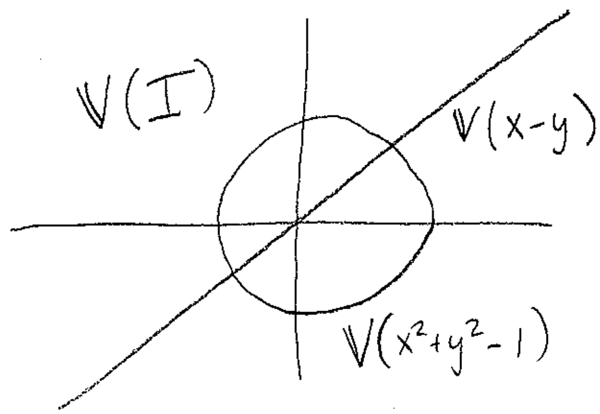
For a variety V, set

$\mathbb{I}(V) = \{f \in k[x_1, \ldots, x_n] \mid f(a) = 0 \ \forall a \in V\}$

Last time: $\mathbb{I}(\mathbb{V}(I)) \supseteq \mathrm{rad}(I) \supseteq I$

equal if k is alg. closed.

Ex: $I = (x^3 - xy^2 - yx^2 - y^3 - x + y) \subseteq \mathbb{R}[x, y]$



Def: A variety V is irreducible if whenever $V = V_1 \cup V_2$ for varieties V_i then $V = V_1$ or $V = V_2$.

Thm: V is irreducible iff $\mathbb{I}(V)$ is prime.

Pf: ($\Rightarrow$) Suppose $f_1 \cdot f_2 \in \mathbb{I}(V)$

Let $V_i = \{$ pts of V where $f_i = 0\}$

$= V \cap \mathbb{V}(f_i) = \mathbb{V}(I + (f_i))$

if $V = \mathbb{V}(I)$.

If $a \in V$ have $0 = (f_1 f_2)(a) = f_1(a) f_2(a)$

$\Rightarrow$ either $f_1(a) = 0$ or $f_2(a) = 0$. So

$$V = V_1 \cup V_2.$$

As V is irreducible, have, say, $V = V_1$.

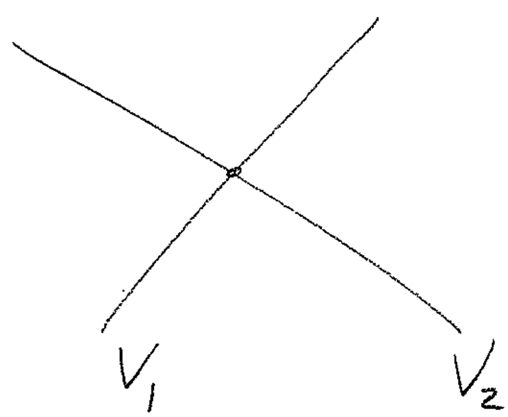
Then $f_1(a) = 0$ for all $a \in V \Rightarrow f_1 \in \mathbb{I}(V)$.

($\Leftarrow$) Suppose $V = V_1 \cup V_2$ with $V_1 \neq V$

As $V_1 \subsetneq V$ have

$\mathbb{I}(V_1) \supsetneq \mathbb{I}(V)$, and so

let $f_1 \in \mathbb{I}(V_1) \setminus \mathbb{I}(V)$.



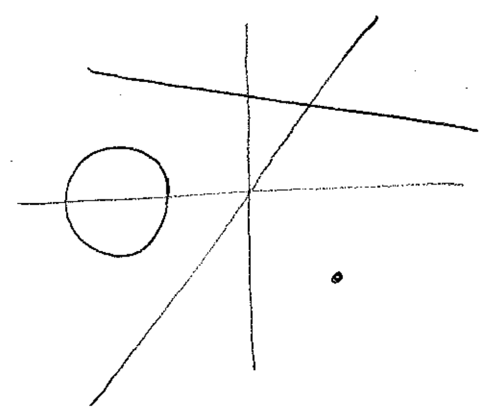
Let $f_2 \in \mathbb{I}(V_2)$. Then $f_1 f_2 = 0$ on $V \Rightarrow f_1 f_2 \in \mathbb{I}(V)$. As $\mathbb{I}(V)$ is prime, one $f_i \in \mathbb{I}(V)$ which must be f_2. So $\mathbb{I}(V_2) \subseteq \mathbb{I}(V) \Rightarrow V_2 \supseteq V$

$\Rightarrow V_2 = V$. ∎

<u>Thm</u>: Any alg. variety $V \subseteq k^n$ is a finite union of irreducible varieties

<u>Suppose not</u>: $V = V_1 \cup W_1$ with $V_1, W_1 \neq V$

If V_1 and W_1 are irred, done.



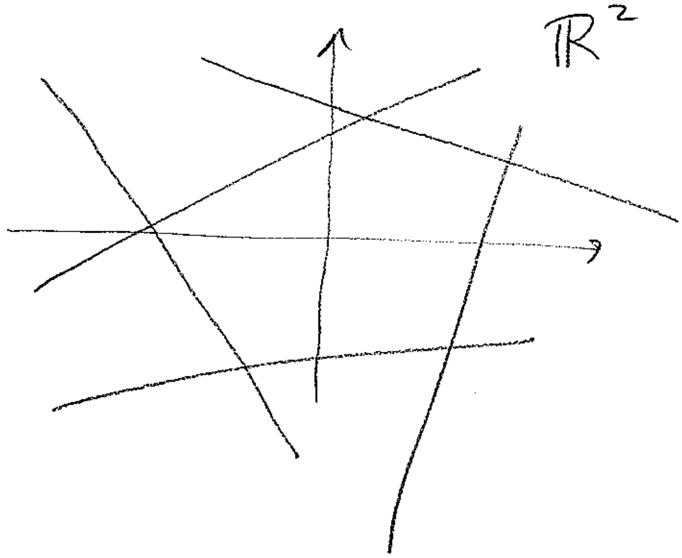
Else, say

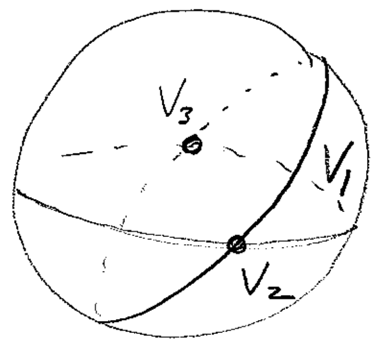
$$V_1 = V_2 \cup W_2 \text{ with}$$

$V_1 \neq V_2 \cup W_2$. Continuing, can construct nonempty varieties

$$V_0 \supsetneq V_1 \supsetneq V_2 \supsetneq V_3 \supsetneq \cdots$$

Doesn't mesh with





In $k[x_1, \dots, x_n]$ if $I_k = \mathbb{I}(V_k)$ have

$$I_0 \subsetneq I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$$

Doesn't happen as $k[x_1, \dots, x_n]$ is

<u>Noetherian</u>, i.e. every seq

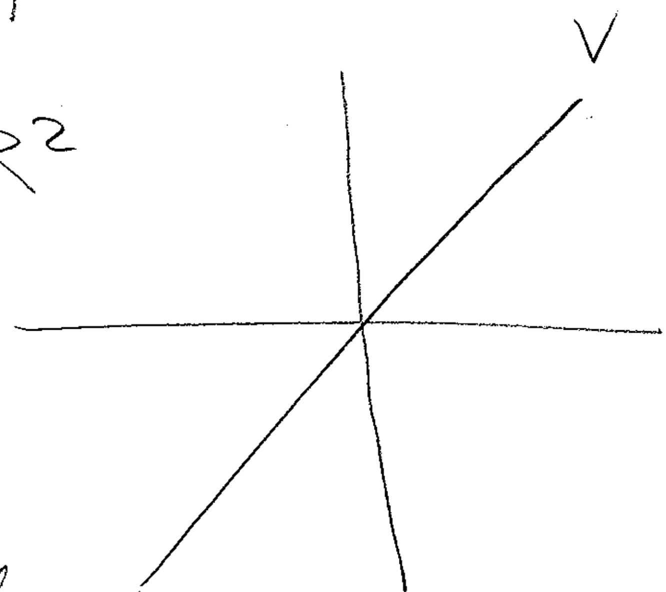
$$I_0 \subseteq I_1 \subseteq \cdots \subseteq I_k \subseteq$$

eventually stabilizes, i.e. $\exists\, n$ s.t. $I_k = I_n$ for all $k = n$.

This is <u>Hilbert's Basis Thm</u>.

$V \subseteq k^n$ a variety

Consider polynomial fns $f: V \to k$.



Ex: $V = \mathbb{V}(x-y)$ in $\mathbb{R}^2$

$f = x^2y + y + x$

$g = y^2x + 2y$

While these have diff. formulas,
we have $f|_V = g|_V = x^3 + 2x$.

since if $(a_1, a_2) \in V$ then $a_1 = a_2$.

Def: The coordinate ring of V is

$$k[V] = \left\{ f: V \to k \;\middle|\; \begin{array}{l} f \text{ is the restriction of} \\ \text{an elt of } k[x_1, \ldots, x_n] \end{array} \right\}$$

Note

$$k[V] = k[x_1, \ldots, x_n] / \mathbb{I}(V)$$

since $\mathbb{I}(V)$ is the kernel of the ring homomorphism:

$$k[x_1, \ldots, x_n] \twoheadrightarrow k[V]$$

Cor: V is irred. iff $k[V]$ is an integral domain.