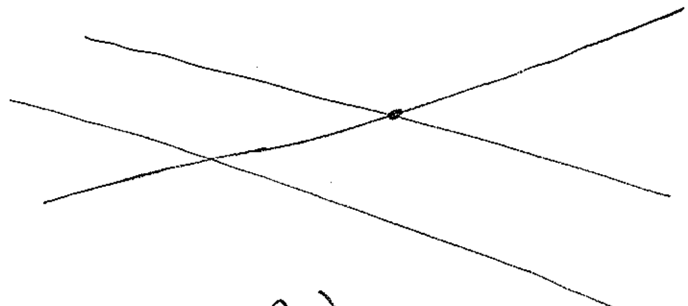
Lecture 34: Projective Space

Inconveniences:

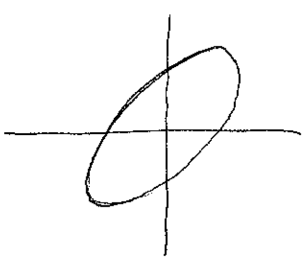
① If $L_1 \neq L_2$ are two lines in $\mathbb{R}^2$, they usually intersect in one pt, except when they don't, i.e. are parallel

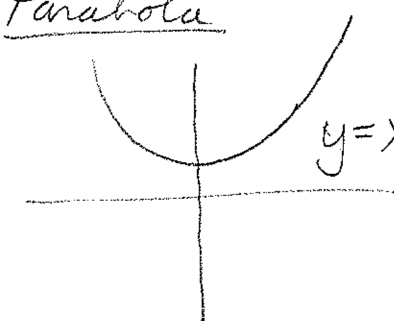


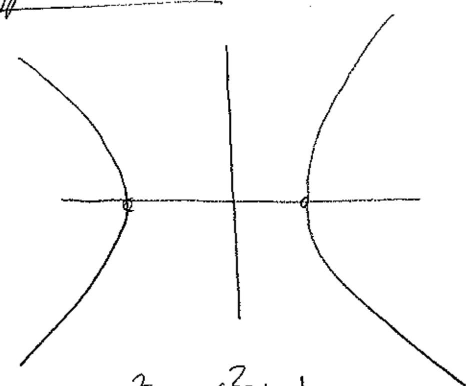
② Plane Conics in $\mathbb{R}^2$:

$$V = \mathbb{V}(ax^2 + bxy + cy^2 + dx + ey + f)$$

Ellipse: Parabola Hyperbola

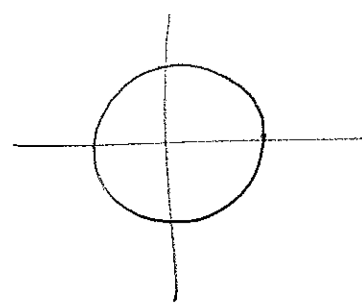






Why can't they are be the same?

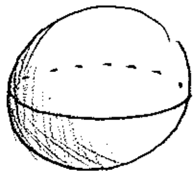
③ $\mathbb{V}(x^2 + y^2 - 1) \subseteq \mathbb{R}^2$ is



What is

$\mathbb{V}(x^2 + y^2 - 1) \subseteq \mathbb{C}^2$?

It should have $\dim_{\mathbb{C}} = 1$ and $\dim_{\mathbb{R}} = 2$,

so one guess is $V =$  .

But V is <u>not</u> compact, as if we fix any $a \in \mathbb{C}$, can solve $a^2 + y^2 = 1$ to find a point $(a, b) \in V$. So V is <u>not</u> bounded in $\mathbb{C}^2 \cong \mathbb{R}^4$.

<u>The fix</u>: Projective Space.

Let k be a field, then

$$\mathbb{P}^n_k = \underset{\uparrow \text{Affine space}}{k^n} \cup \left\{ \begin{matrix} \text{stuff at} \\ \infty \end{matrix} \right\}$$

Consider $\mathbb{P}^2_{\mathbb{R}}$, the projective plane. Need to add points at ∞ s.t.

ⓐ Any two parallel lines meet at ∞.

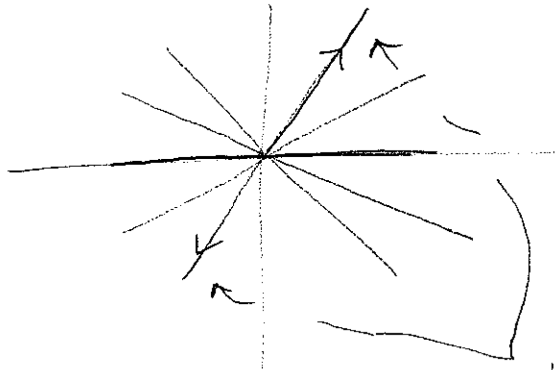
ⓑ Any two nonparalle lines <u>don't</u> meet at ∞.

Suggests: Set $S = \mathbb{P}^2_{\mathbb{R}} \setminus \mathbb{R}^2$

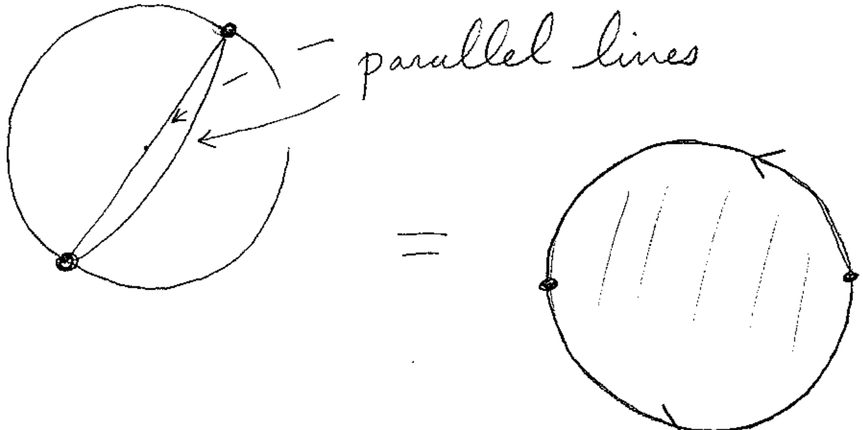
One pt in S for each line L through 0.

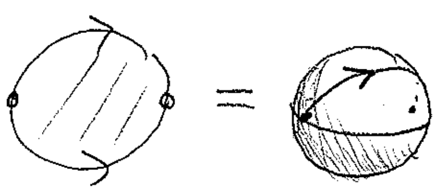
Note S is a circle

Hence $\mathbb{P}^2_{\mathbb{R}}$ looks like



to the same pt in S



[Compare ]

<u>Def</u>:

$$\mathbb{P}^2_{\mathbb{R}} = \{\text{lines through } 0 \text{ in } \mathbb{R}^3\}$$

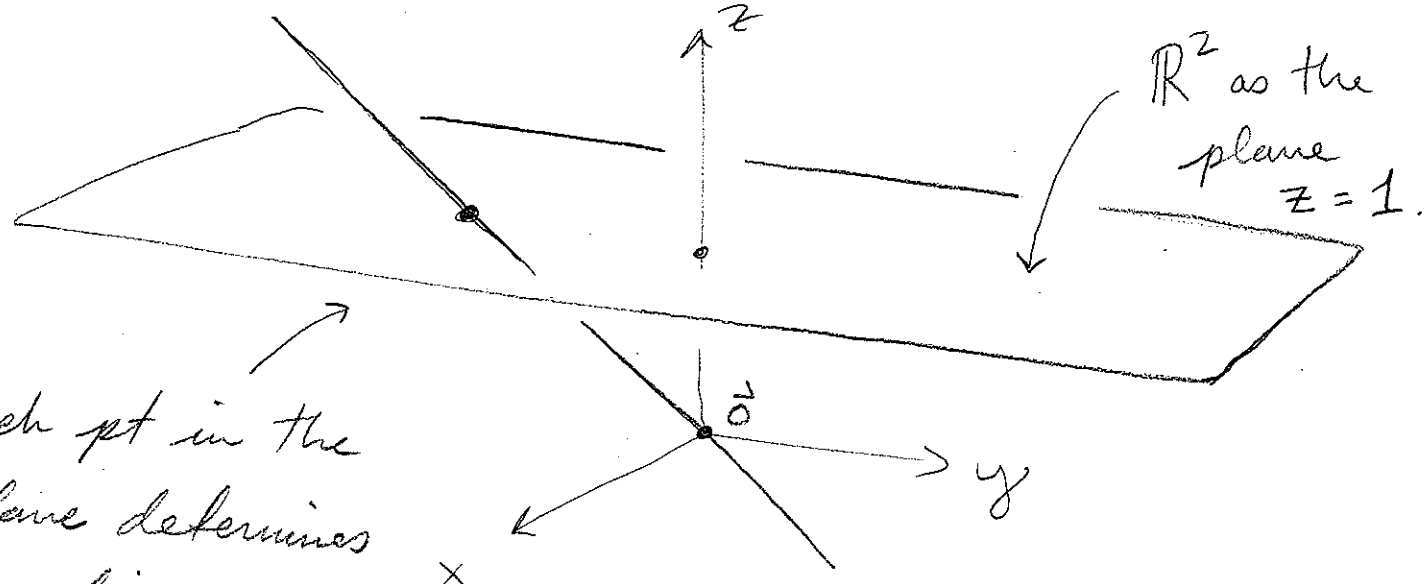
$$= \{(x,y,z) \in \mathbb{R}^3 \setminus \{0\}\} \Big/ (x,y,z) \sim (\lambda x, \lambda y, \lambda z)$$

for $\lambda \notin \mathbb{R}^\times$

Denote pt in $\mathbb{P}^2_{\mathbb{R}}$ by $(x:y:z)$

<u>Observations</u>:

$\mathbb{R}^2 \subseteq \mathbb{P}^2_{\mathbb{R}}$ as $\{(x:y:1)\}$ a pt has at most one rep. of this form.



Each pt in the plane determines a a line through $\vec{o}$.

What's at ∞?

$$\mathbb{P}^2_{\mathbb{R}} \setminus \mathbb{R}^2 = \{(\underbrace{x : y}_{\text{not both } 0} : 0)\}$$

$$= \{\text{lines through } 0 \text{ in } \mathbb{R}^2\} = \mathbb{P}^1_{\mathbb{R}}$$

For any k, n can define

$$\mathbb{P}^n_k = \{\text{lines through } 0 \text{ in } k^{n+1}\}$$

$$= \{a \in k^n \setminus \{0\}\} \big/ a \sim \lambda a \text{ for } \lambda \in k^\times$$

which contains $k^n = \mathbb{A}^n_k$ as $\{(a_1 : \cdots : a_n : 1)\}$

with $\mathbb{P}^n_k \setminus \mathbb{A}^n_k = \mathbb{P}^{n-1}_k$

Ex: $\mathbb{P}^1_k = k \cup \{(1:0)\}$

$\leftarrow$ the pt at ∞.

$\mathbb{P}^1_{\mathbb{C}} =$ 

the Riemann sphere.

$V \subseteq k^n$ will now be called *affine varieties*

in contrast to *projective varieties*:

Whats that? How can we make sense of a poly eqn on $\mathbb{P}^2_{\mathbb{R}}$? Want to write polys using coor $(x:y:z)$ but its not well-def:

$$f = xy - z \qquad \begin{aligned} f(1:1:1) &= 0 \\ f(2:2:2) &= 2. \end{aligned}$$

Def: A poly $f \in \mathbb{R}[x,y,z]$ is homogenous if the degree ($x^a y^b z^c \to a+b+c$) of all terms are the same.

Ex: $xy - z^2$ $\qquad$ Non-Ex $xy - z$

Suppose $f \in \mathbb{R}[x,y,z]$ is homogenous.

Consider

$$\mathbb{V}(f) = \left\{ (a:b:c) \in \mathbb{P}^2_{\mathbb{R}} \;\middle|\; f(a,b,c) = 0 \right\}$$

which makes sense because for $\lambda \neq 0$, we have

$$f(\lambda a, \lambda b, \lambda c) = \lambda^n f(a,b,c)$$

where n is the degree of f.

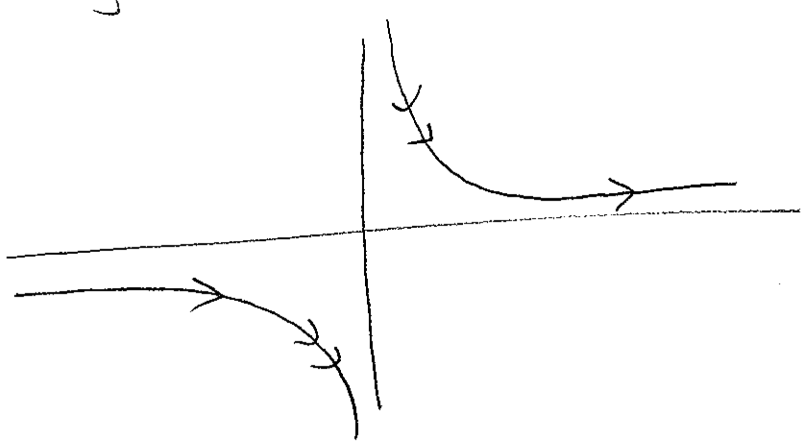
<u>Ex</u>: $V = \mathbb{V}(xy - z^2)$.

What is $V \cap \mathbb{R}^2$?

$$= \{ (x:y:1) \mid xy - 1 = 0 \} = \mathbb{V}_{\mathbb{R}^2}(xy-1)$$

What is $V \cap \mathbb{P}^1_{\mathbb{R}}$

$$= \{ (x:y:0) \mid xy = 0 \} = \{ (1:0:0), (0:0:1) \}$$



<u>Note:</u>

$V \cong$