Lecture 36: Curves over $\mathbb{C}$

Last time: Plane conics in $\mathbb{P}^2_{\mathbb{R}}$.

Have $\mathbb{V}_{\mathbb{R}^2}(x^2+y^2-1) =$ . The last thing introducing projective space was to fix was the fact that $\mathbb{V}_{\mathbb{C}^2}(x^2+y^2-1) \neq$ .

Q: What is $\mathbb{V}_{\mathbb{P}^2_{\mathbb{C}}}(x^2+y^2-z^2) = V$?

A: Last time, saw how all non-degen real conics are the same, so we can just as well consider

$$V' = \mathbb{V}_{\mathbb{P}^2_{\mathbb{C}}}(x^2-yz)$$

which you can check is $P_A(V)$ with

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Then

at ∞.

$$V' = \mathbb{V}_{\mathbb{C}^2}(x^2-y) \cup \{(0:1:0)\}$$

param. by x, so $= \mathbb{C}$

So $V' = \mathbb{C} \cup \{pt\} =$  .

Explicitly, have

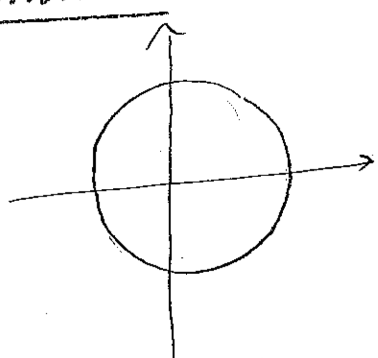
$$\mathbb{P}^1_{\mathbb{C}} \xrightarrow{\cong} V'$$

$$(u:v) \longmapsto (uv : u^2 : v^2)$$

well-defined as all terms are quadratic.

Let k be a field. An affine variety $V = \mathbb{V}(f) \subseteq k^2$ is <u>nonsingular</u> or <u>smooth</u> if $Df = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ is never 0 at any point of V

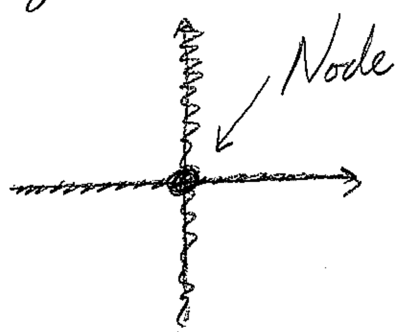
<u>Smooth</u>:



$\mathbb{V}(x^2+y^2-1)$

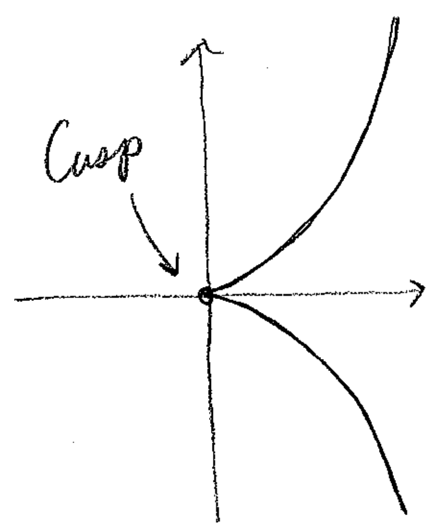
$Df = (2x, 2y)$

<u>Singular</u>:



$\mathbb{V}(xy)$

$Df = (y, x)$.



$\mathbb{V}(y^2 - x^3)$

$Df = (-3x^2, 2y)$

When $k = \mathbb{R}$ or $\mathbb{C}$, the Implicit Fn. Thm tells us that a non-singular variety of this kind looks locally like $\mathbb{R}$ or $\mathbb{C}$.

Such varieties are called <u>curves</u>.

For $C = \mathbb{V}_{\mathbb{P}^2_k}(f)$, say it is smooth if each of the affine curves

$$C \cap \{(x:y:1)\},\ C \cap \{(1:y:z)\},\ C \cap \{(x:1:z)\}$$

is smooth.

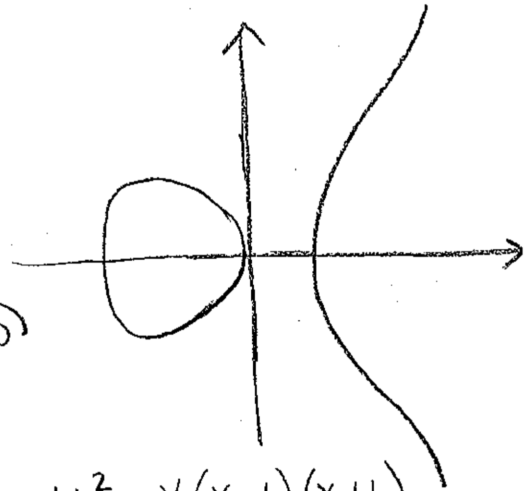
Next after lines and conics are <u>elliptic curves</u>, e.g.

$$C = \mathbb{V}_{\mathbb{P}^2_k}\left(y^2 - x\,(x^2 + ax + b)\right)$$

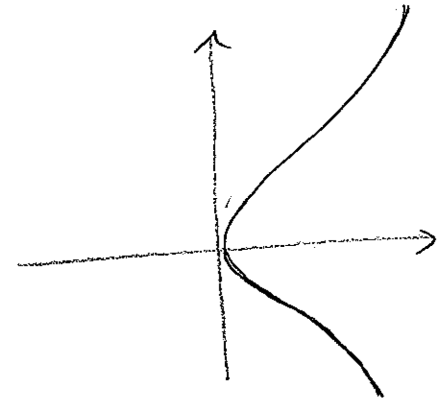
[<u>Aside</u>: When $k = \mathbb{R}$ or $\mathbb{C}$, any curve coming from a cubic equation can be put in this form via a projective transformation]

Ex: $k = \mathbb{R}$

In $\mathbb{P}^2_{\mathbb{R}}$, both also have one pt at ∞, namely $(0:1:0)$



$y^2 = x(x-1)(x+1)$



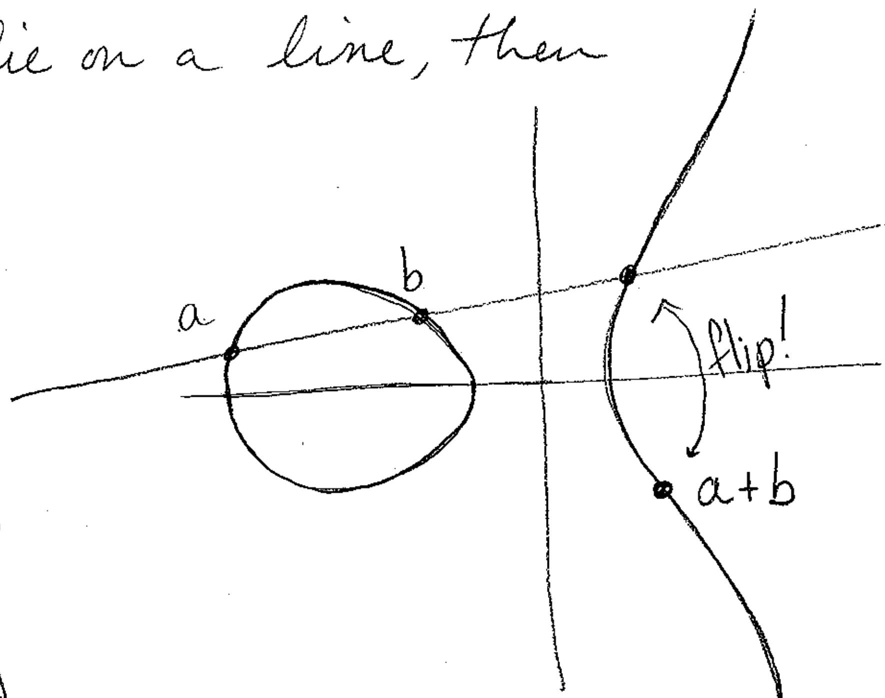
$y^2 = x(x^2+1)$

Suppose C is smooth. Then we can make it into an abelian gp, where

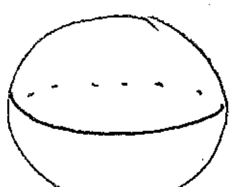
(a) $\mathcal{O} = (0:1:0)$ is the ident elt.

(b) The inverse of (x,y) is $(x,-y)$.

(c) If $a,b,c \in C$ lie on a line, then $a+b+c = \mathcal{O}$



Q: Suppose $k = \mathbb{C}$.

Is $C =$  as with a line or conic?

Consider the map $\pi : C \to \mathbb{P}^1_{\mathbb{C}}$.

$$(x:y:z) \longrightarrow (x:z)$$

That is, on $C \cap \mathbb{C}^2$, the map π is just proj $(x,y) \to x$ and the point at ∞ in C goes to the pt at ∞ in $\mathbb{P}^1_{\mathbb{C}}$. If C is def by

$$y^2 = x(x-\alpha)(x-\beta) \qquad (*)$$

then

Claim: π is generically 2-to-1. In particular,

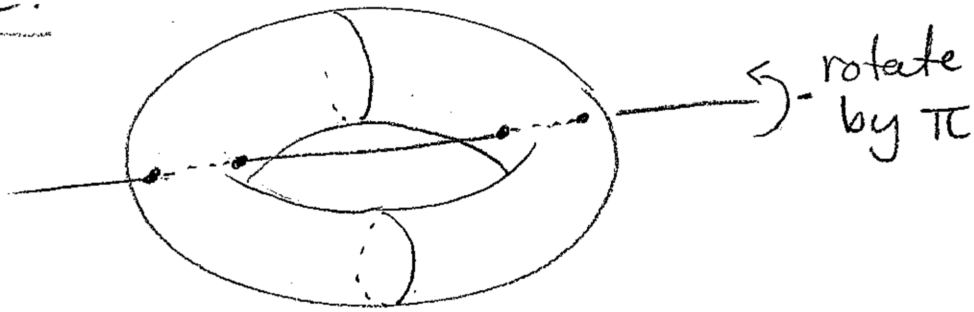
$$\pi^{-1}(p) = \text{two pts except when } p \in \{0, \alpha, \beta, \infty\}$$

Pf: One x is fixed, $(*)$ has two sol'ns unless the RHS vanishes, in which case there's only one.

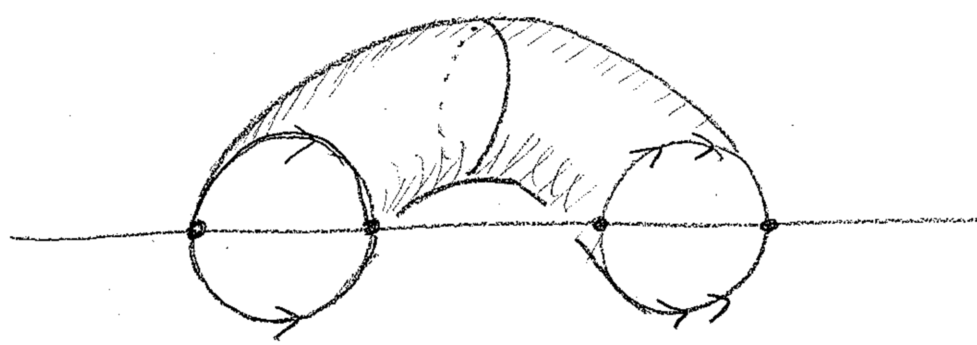
The symmetry of C given by $y \to -y$ respects π, and so $C/_{y \sim -y} = \mathbb{P}^1_{\mathbb{C}}$.

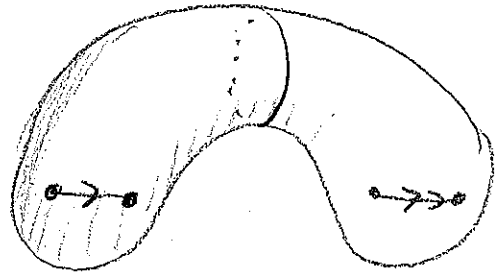
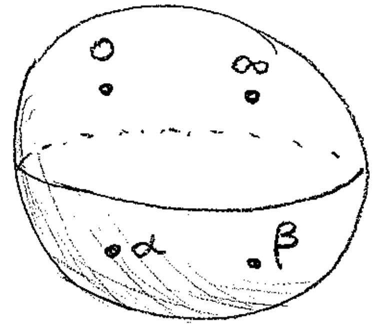
Geometric Picture:

What is the quotient here?



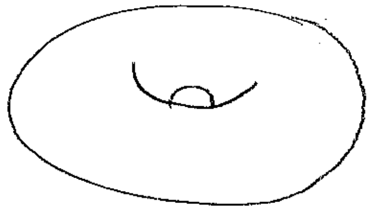
Each pt is equiv to one on the back half:



$=$  $=$ 

Turns out this is exactly the picture of $C \to \mathbb{P}^1_{\mathbb{C}}$!

Why is this plausible? Well,

for one thing  $= S^1 \times S^1$

$= \mathbb{R}^2/\mathbb{Z}^2$ is a group, like $\mathbb{C}$.

Also, locally π is a homeomorphism,

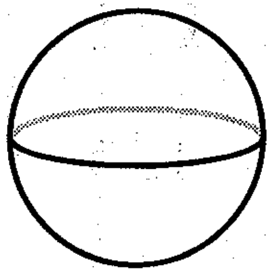
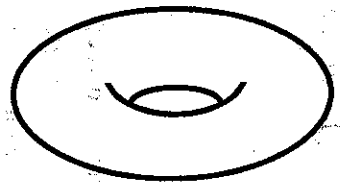
except at $(0,0)$, $(0,\alpha)$, $(0,\beta)$, ∞

where it looks like $\underset{z \longrightarrow z^2}{\mathbb{C} \to \mathbb{C}}$. called

"A two-fold cover of $\mathbb{P}^1_{\mathbb{C}}$ branched at 4-pts"

Turns out this is the only such cover...

If time remains, discuss some of:

	$g = 0$	$g = 1$	$g \geq 2$
Topology C is homeomorphic to:			
fundamental group:	simply connected	$\pi_1 = \mathbb{Z} \oplus \mathbb{Z}$	like free group on $2g$ generators
Algebraic/complex analytic geometry embeddings, concrete descriptions:	$C \cong \mathbb{P}^1_{\mathbb{C}}$ $\cong C_2 \subset \mathbb{P}^2_{\mathbb{C}}$	$C \cong C_3 \subset \mathbb{P}^2_{\mathbb{C}}$ $\cong \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\cdot\tau)$	no simple description, but e.g. most curves of genus 3 are $\cong$ nonsing. $C_4 \subset \mathbb{P}^2_{\mathbb{C}}$
automorphisms:	3-dimensional group of projective transformations	translations in group law × finite group	finite group
moduli:	none	1 modulus (cross-ratio or j-invariant)	$3g-3$ moduli
Differential geometry there exists a natural class of Riemannian metrics with constant curvature:	constant positive curvature	zero curvature (that is, flat)	constant negative curvature
Diophantine problems if $k = \mathbb{Q}$ or a number field (that is, $[k : \mathbb{Q}] < \infty$) then:	$C_k = \emptyset$ or $\mathbb{P}^1_k$	C_k is a finitely generated Abelian group (Mordell-Weil theorem)	C_k is a finite set (Faltings' Theorem, Mordell conjecture)