Lecture 38

Last time:

$V \subseteq K^n$ an <u>irreducible</u> affine variety

<u>Function Field</u> $K(V)$: field o fracs of $K[V]$.

For $f \in K(V)$, called a <u>rational function</u>, set

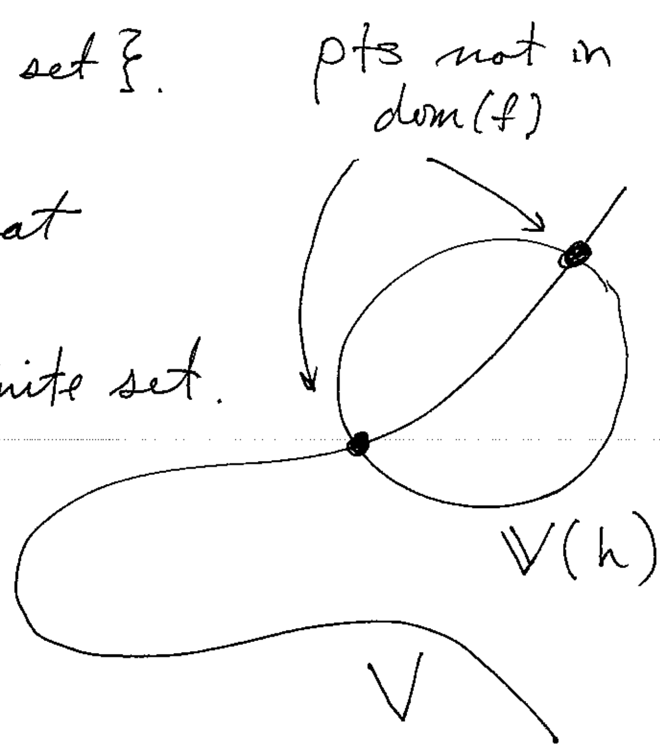
$$\mathrm{dom}(f) = \left\{ p \in V \,\middle|\, \begin{matrix} \text{can rep } f \text{ as } \frac{g}{h} \text{ with} \\ h(p) \neq 0 \end{matrix} \right\}$$

<u>Prop</u>: Suppose $V = \mathbb{V}_{\mathbb{C}^2}(p)$ is a smooth plane curve. Then for any $f \in \mathbb{C}(V)$,

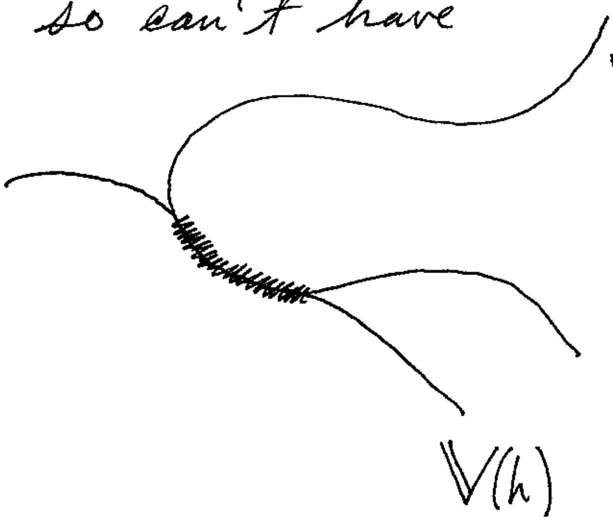
$$\mathrm{dom}(f) = V \setminus \{\text{finite set}\}.$$

<u>Idea</u>: If $f = \frac{g}{h}$ want that

$$V' = \mathbb{V}(p, h) = \text{finite set}.$$



<u>Moral</u>: Polys are det. locally, not too complicated, so can't have



or

<u>Similar Fact</u>: $f, g \in \mathbb{C}[z]$. If $\exists\, \varepsilon > 0, z_0$ s.t. $f(z) = g(z)$ for all $z \in B_\varepsilon(z_0)$, then $f = g$.

<u>Pf</u>: If $f = g$ on $B_\varepsilon(z_0)$, then $f^{(n)}(z_0) = g^{(n)}(z_0)$ for all n. Thus $f = g$ as elts of $\mathbb{C}[z]$ by looking at the Taylor series. $\blacksquare$

<u>Comforting fact</u>: If V is a smooth irred. plane curve in $\mathbb{C}^2$, then any $f \in \mathbb{C}(V)$ comes from an honest continuous function

$$\tilde{f} : V \to \mathbb{P}^1_{\mathbb{C}} \quad \text{where } \tilde{f}(p) = \infty \iff p \notin \mathrm{dom}(f)$$

Ex: $V = \mathbb{C}$, then $f \in \mathbb{C}(V) = \mathbb{C}(t)$ has the form
$$f = c\,\frac{(t-a_1)\cdots(t-a_k)}{(t-b_1)\cdots(t-b_\ell)} \quad \text{with}$$

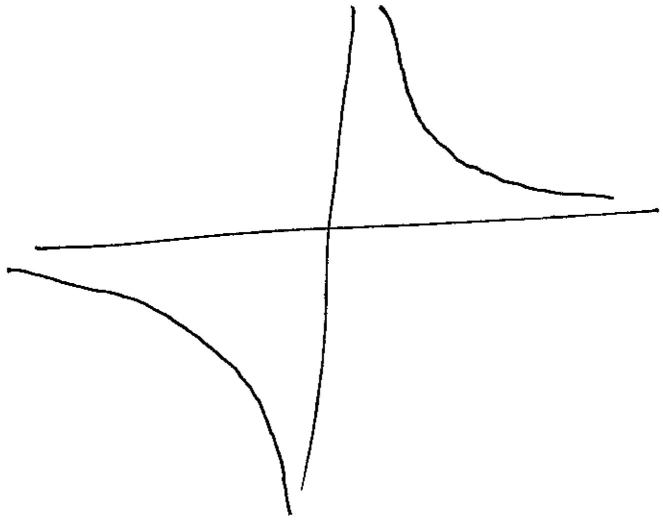
$\mathrm{dom}(f) = \mathbb{C} \setminus \{b_1, \dots, b_k\}$.

[Our goal here is something about solving the inverse Galois prob for $\mathbb{C}(t)$...]

Let V be a smooth irred. plane curve in $\mathbb{C}^2$, and $h \in \mathbb{C}[V]$ be a polynomial. If we regard h as
$$h : V \longrightarrow \mathbb{C}$$
it induces a ring homomorphism
$$\mathbb{C}[V] \xleftarrow{h^*} \mathbb{C}[t] = \mathbb{C}[\mathbb{C}]$$

~~...~~ via
$$h^*(f) = f \circ h, \text{ that is } h^*(f)(x,y) = f(h(x,y)).$$

<u>Ex</u>: $V = \mathbb{V}(xy-1)$



$h = x + y \in \mathbb{C}[V]$

$f(t) = t^2 + t + 1 \in \mathbb{C}[t]$

$= (x+y)^2 + (x+y) + 1$

$h^*(f) = (x^2 + 2xy + y^2) + (x+y) + 1$

$= x^2 + y^2 + x + y + 3$ in $\mathbb{C}[V]$.

In general, what is $\ker(h^*)$? Suppose $f \in \mathbb{C}[t]$ is $\neq 0$. If $h^*(f) = 0$, then $f(h(x,y)) = 0$ in $\mathbb{C}[V] \Rightarrow$ Every pt in $h(V)$ is a root of $f \Rightarrow h(V)$ is finite

$\Rightarrow$ (as V is irreducible) $h(V) =$ one pt $\Rightarrow h$ is a constant map, coming from a poly w/ only a const term.

<u>Prop</u>: If h is not constant, then $\ker(h^*) = 0$.

Now define a field homom

$$h^*: \mathbb{C}(t) \longrightarrow \mathbb{C}(V)$$

by

$$h^*\left(\frac{p(t)}{q(t)}\right) = \frac{h^*(p(t))}{h^*(q(t))} = \frac{p(h(x,y))}{q(h(x,y))}$$

<u>Note</u>: As long as h is nonconst, this is well defined as $q(t) \neq 0 \Rightarrow h^*(q(t)) \neq 0$ by the Prop.

As $h^*(1) = 1 \neq 0$, the field homom is non-trivial and hence 1-1; so we have

$$\mathbb{C}(t) \overset{}{\underset{h^*}{\hookrightarrow}} \mathbb{C}(V)$$

i.e. $\mathbb{C}(V) / \mathbb{C}(t)$ is an extension of fields. <u>Next time</u>: An example...