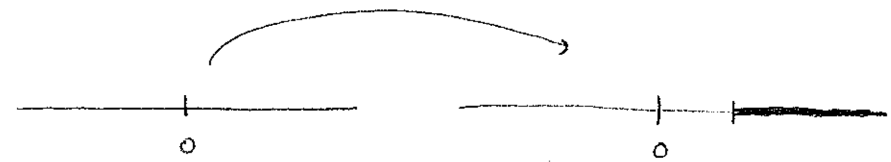
Lecture 16: Proof of the fund. thm of algebra.

Thm: Every non-const $p(z) \in \mathbb{C}[z]$ has a root in $\mathbb{C}$.

Cor: $p(z) \in \mathbb{C}[x]$ non-constant. Then $p: \mathbb{C} \to \mathbb{C}$ is onto.

Pf: Given $w \in \mathbb{C}$, the poly $f(z) = p(z) - w$ has a root. ▨

Notes: ① Plenty of $p(x) \in \mathbb{R}[x]$ don't have roots in $\mathbb{R}$, e.g. x^2+2.



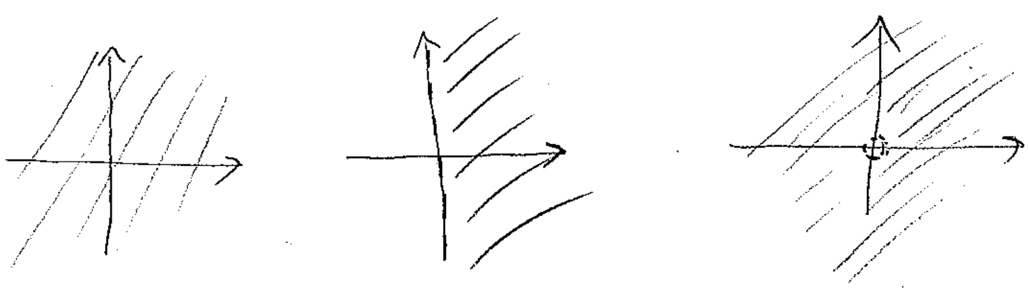
② $p: \mathbb{C} \to \mathbb{C}$ is a very nice fn from $\mathbb{R}^2$ to $\mathbb{R}^2$, but many such aren't onto, e.g. $(x,y) \longmapsto (x^2+y^2, xy-1)$

misses $(0,0)$

Q: Whats so special about poly maps from $\mathbb{C}$ to $\mathbb{C}$??

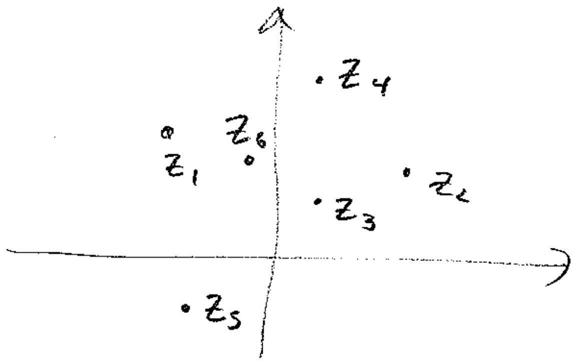
① $p(\mathbb{C})$ is a <u>closed</u> subset of $\mathbb{C}$, in contrast to

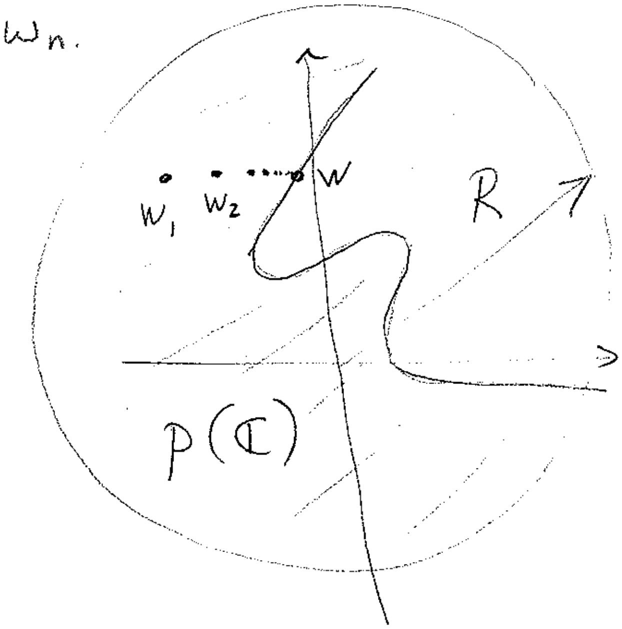
$$\mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad \text{or} \qquad \mathbb{C} \to \mathbb{C}$$
$$(x,y) \longrightarrow (e^x, y) \qquad\qquad z \longrightarrow e^z$$



Proof: Suppose $\{w_n\} \subseteq p(\mathbb{C})$ converge to w.

Let z_n be such that $p(z_n) = w_n$.





Now

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \quad + a_1 z + a_0$$

and so when z is large, $|p(z)| > R$. So

all $z_i \subseteq B_0(R')$. So some subseq $z_{n_k} \to z_0$ in $\mathbb{C}$.

Then

$$p(z_0) = p\left(\lim_{k\to\infty} z_{n_k}\right) \overset{\text{cont of } p}{=} \lim_{k\to\infty} \underbrace{p(z_{n_k})}_{w_{n_k}} = w.$$

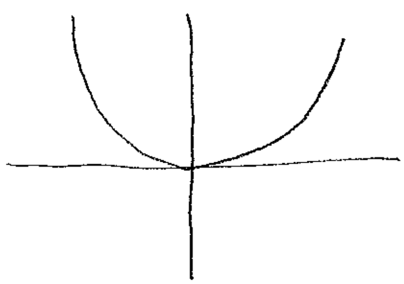
$\blacksquare$

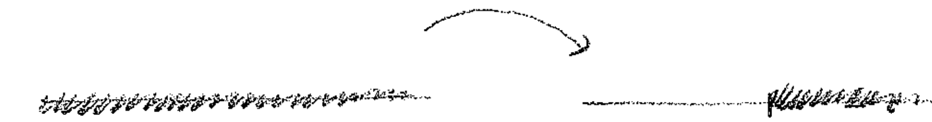
<u>Shorter NMD:</u> Image of cpt is cpt.

[Note that ① holds for $\mathbb{R}$ too, so this is only part of the story.]

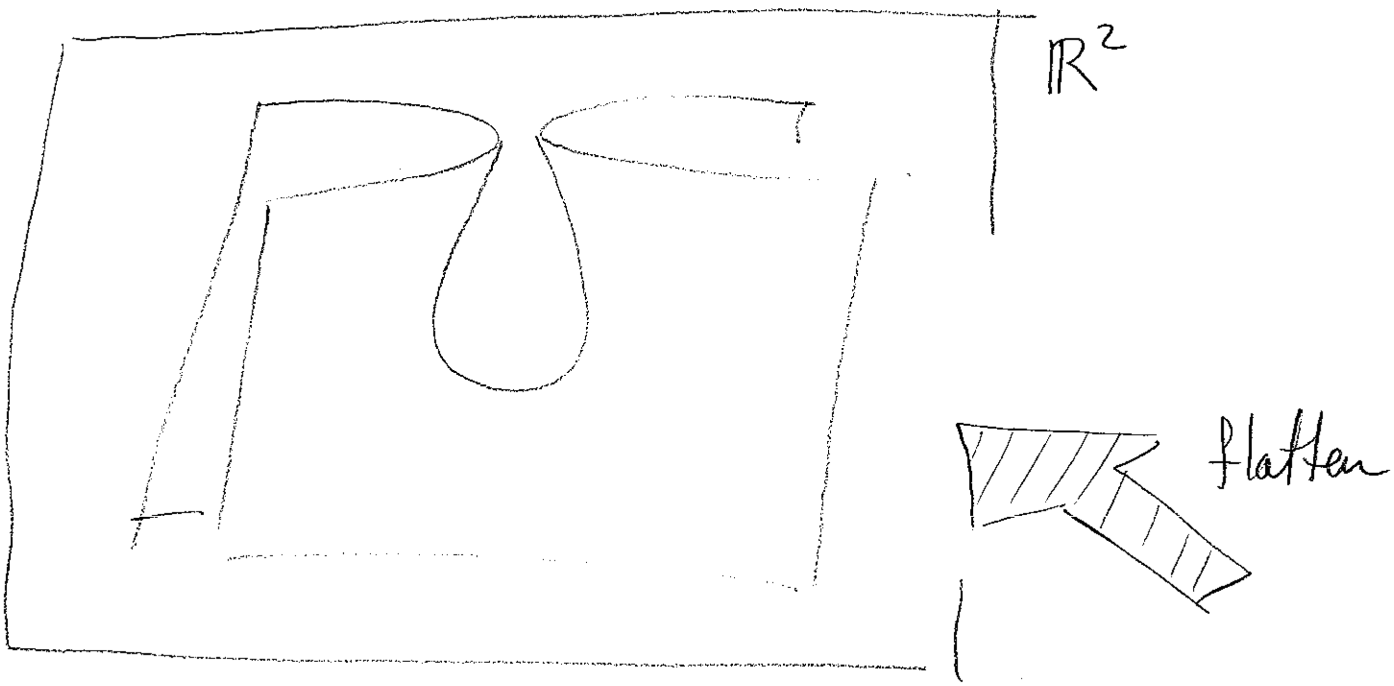
$p : \mathbb{R} \to \mathbb{R}$ not onto because it folds.

$x \longrightarrow x^2$





Could have a similar problem for $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$



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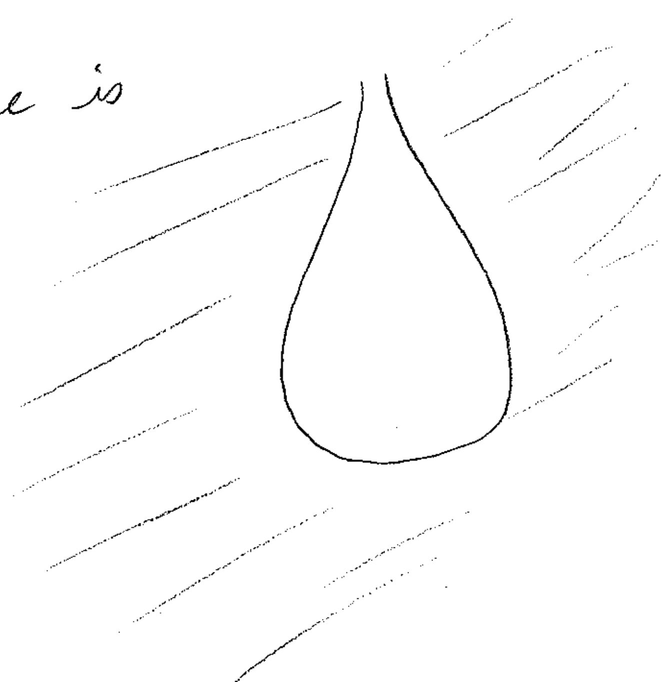
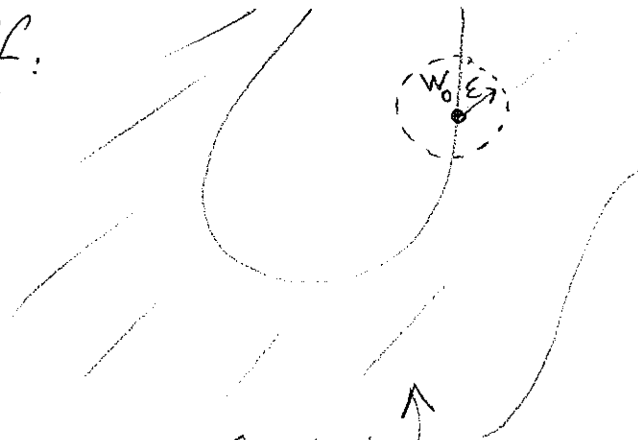


Key: complex $p(z)$ can't "fold".

Claim: Suppose $p(z)$ is not onto as a map from $\mathbb{C}$ to $\mathbb{C}$.

Then $\exists\, w_0 \in p(\mathbb{C})$ s.t. no $\underbrace{B_\varepsilon(w_0)}_{= \{ z \mid |z - w_0| < \varepsilon \}} \subseteq p(\mathbb{C})$.

Pf:



Take $w_1 \notin p(\mathbb{C})$ and $w_2 \in p(\mathbb{C})$, and

Looking for this.

consider the line segment between them. Take w to be the closed pt in $p(\mathbb{C}) \cap L$ to w_0.

(exists since $p(\mathbb{C})$ is closed). ▨

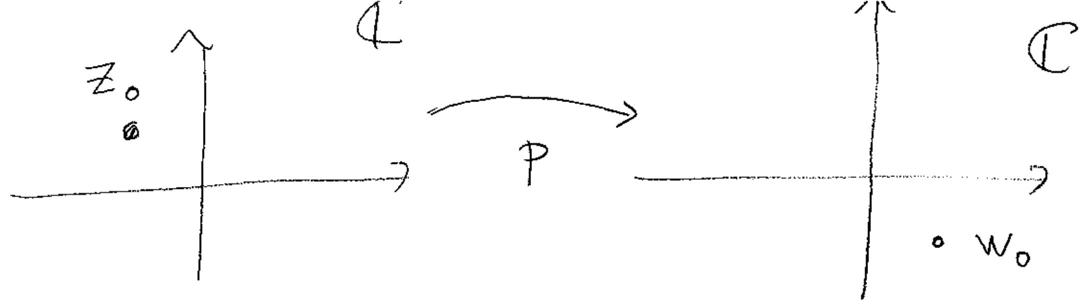
Addendum: Can choose w so that $p(z_0) = w$ and $p'(z_0)$ is non-zero. $\left[\begin{array}{l} \text{only finitely many pts} \\ \text{where } p'(z_0) = 0. \end{array} \right]$

The FTA now follows from.

Lemma 1: $p(z) \in \mathbb{C}[z]$. Suppose $p'(z_0) \neq 0$.

Then $\exists\, \varepsilon > 0$ so that $p(\mathbb{C}) \supseteq B_\varepsilon(p(z_0))$.

Proof:



Change coordinates so $z_0 = w_0 = 0$.

(i.e. replace p by $p(z+z_0) - w_0$)

Now, consider

$$p(z) = a_1 z + a_2 z^2 + \cdots + a_n z^n \qquad a_i \in \mathbb{C}$$

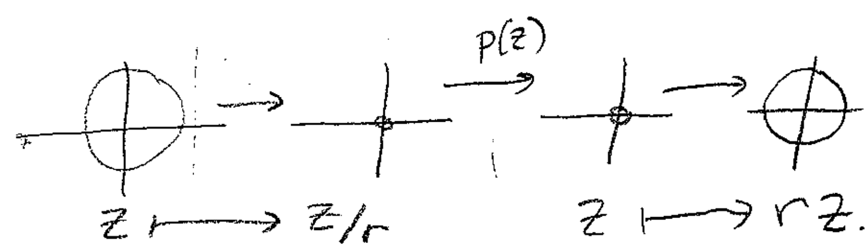
chang coor. on image $\mathbb{C}$ by mult. by $\frac{1}{a_1} = re^{i\theta}$

So now

$$p(z) = z + a_2 z^2 + \cdots + a_n z^n$$

Recall the point of the derivative: fns look

linear on small scales.



So change coor on domain by $z' = \frac{z}{r}$ ($r \leftarrow$ large)

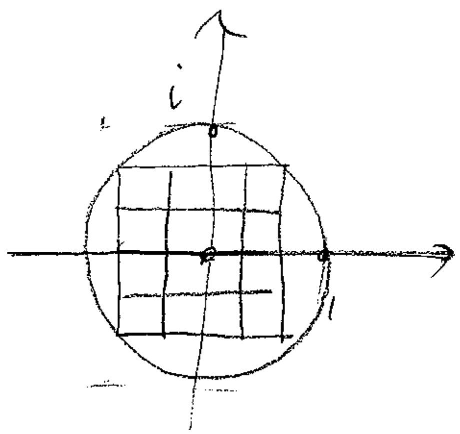
and range by $z' = rz$. Then

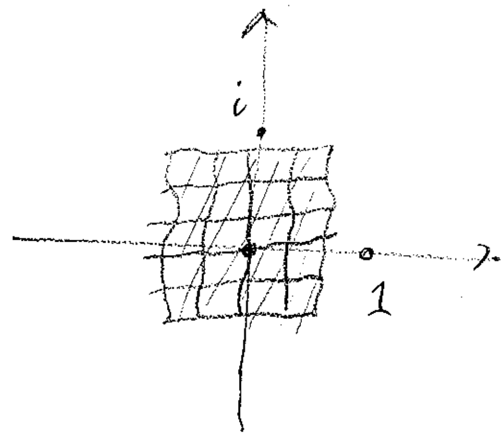
$$P_{new} = r \cdot p\left(\frac{z}{r}\right) = z + \frac{a_2}{r} z^2 + \frac{a_3}{r^2} z^3 + \cdots + \frac{a_n}{r^{n-1}} z^n$$

Thus, assume

$$p(z) = z + b_2 z^2 + b_3 z^3 + \cdots + b_n z^n$$

where b_i are tiny. Now, what does p look like on $B_1(0)$





Say $|b_i| < \frac{1}{n\,10^{10}}$. Then on the disc shown

$$p(z) = z + E(z) \quad \text{where } |E(z)| < 10^{-10}$$

<u>Intuition:</u> Map must be onto near O, as needed.

This is the content of the Inverse Fn. Thm.

<u>Direct proof:</u> Fix w in $B_{1/8}(0)$. Will find z_0 with $p(z_0) = w$. Will use Newton's Method; fix $|b_i| < \delta$

<u>Lemma 2</u>: Suppose $z \in B_1(0)$. If $|p(z) - w| < \varepsilon$ then

$$z' = z - \frac{p(z) - w}{p'(z)}$$

$\leftarrow$ Newton for $f(x) = p(z) - w$

Sat $|p(z') - w| < \varepsilon^2$

and $|z - z'| < 10\,\varepsilon$

<u>Pf of Lemma 1 assuming Lemma 2.</u>

Take $z_1 = w$. Then $|p(z_1) - w| < 10^{-10}$

So get z_2 with $|p(z_2) - w| < 10^{-20}$

and $|z_2 - z_1| < 10^{-9}$. Repeating, get z_3 with $|p(z_3) - w| < 10^{-40}$ and $|z_3 - z_2| < 10^{-19}$.

As $|z_n - z_{n+1}|$ goes like a geom. series, have $z_n \longrightarrow z_0$ and by cont $p(z_0) = \lim p(z_n) = w$.

Proof of Lemma 2: Assume $\delta < \frac{1}{n^2 10^{-10}}$

First, observe $|p'(z)| = |1 + 2b_2 z + \cdots + n b_n z^{n-1}|$

$$\geq 1 - 10^{-10} > \frac{1}{10}.$$

So $|z' - z| = \dfrac{|p(z) - w|}{|p'(z)|} < 10\varepsilon$

Second, suppose $\delta < \dfrac{1}{n!(n^2)10^{-10}}$

$$p(z') = p\Big(z - \overbrace{\frac{p(z)-w}{p'(z)}}^{-a}\Big) = p(z+a)$$

$$= p(z) + p'(z)a + \underbrace{\frac{p''}{2}(z)}_{< \frac{1}{n\,10^{-10}}} a^2 + \cdots + \underbrace{\frac{p^{(n)}}{n!}(z)}_{< \frac{1}{n\,10^{-10}}} a^n$$

$$= p(z) + p'(z)\left(\frac{w - p(z)}{p'(z)}\right) + E$$

$$= w + E$$

where

$$|E| < \frac{1}{10^{-18}}\varepsilon^2$$

as needed.