Lecture 40:

Thm: G a finite group. Then $\exists$ a Galois extension K of $\mathbb{C}(t)$ with $Gal(K/\mathbb{C}(t))$.

Last time: Given an irred curve $V \subseteq \mathbb{C}^2$ a poly fn $h \in \mathbb{C}[V]$ (e.g. proj to the x-axis) get that $K = \mathbb{C}(V)$ is a finite extension of $\mathbb{C}(t)$.

Plan: ① Given G find a curve V in $\mathbb{P}^n_{\mathbb{C}}$ on which G acts via symmetries, so that $V/G = \mathbb{P}^1_{\mathbb{C}} =$ 

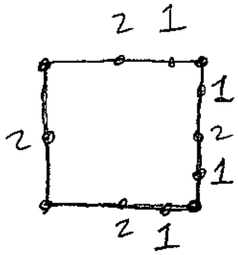
② Each $\sigma \in G$, thought of a sym of V, gives an auto of $K = \mathbb{C}(V)$, via

$$\sigma^*(f) = f \circ \sigma^{-1} \text{ where } f : V \to \mathbb{P}^1_{\mathbb{C}}.$$

is a rat'l fn

[Aside: Check about group actions]

③ $K_G = \mathbb{C}(V)_G = \mathbb{C}(V/G) = \mathbb{C}(\mathbb{P}^1_{\mathbb{C}}) = \mathbb{C}(t)$.

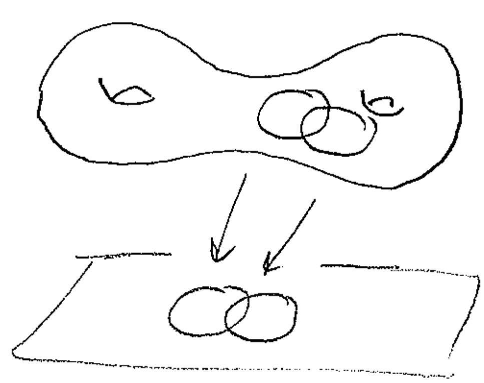
Toy ex: $V =$ graph  $V/G =$  } half of one side

$G = D_4$

Sadly, don't time to prove the whole thing as need a third perspective

$$\left\{ \begin{matrix} \text{Alg. curves} \\ \text{over } \mathbb{C} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{finite extensions} \\ \text{of } \mathbb{C}(t) \end{matrix} \right\}$$

Complex analysis $\supseteq$ {Riemann surfaces}



Also need some topology of covering spaces

Charts!

Instead, I'll do an example with $G = S_3$.

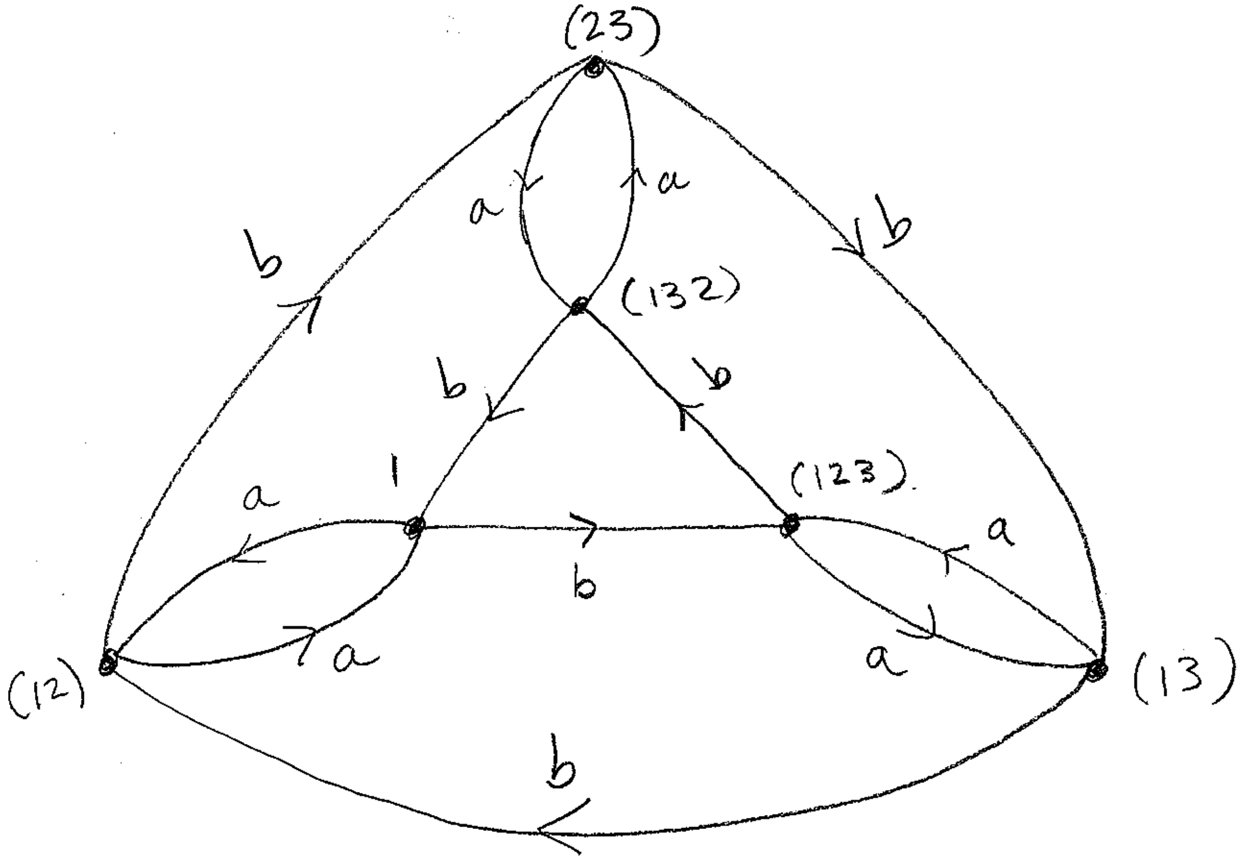
Given a finite group G, let's make it act on some geometric object.

Def: Let S be a generating set for G.
The Cayley Graph $\Gamma(G,S)$ has

1. a vertex v_g for each $g \in G$.
2. an edge labeled s from v_g to v_{gs} for each $g \in G$ and $s \in S$.

Ex: $S_3 = \{1, (12), (13), (23), (123), (132)\}$

$S = \{a = (12),\ b = (123)\}$

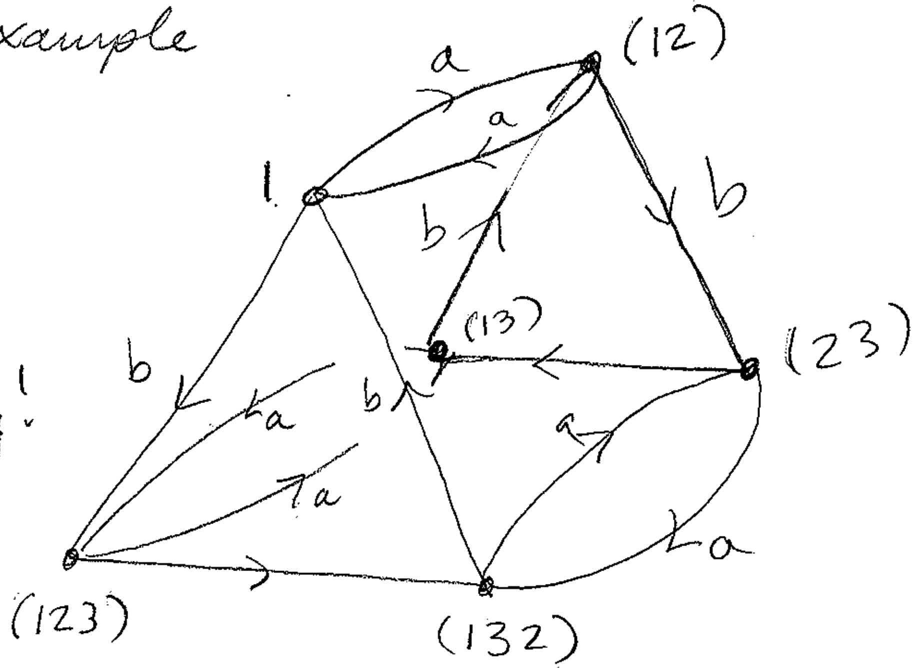


Q: Is $abab^{-1}ab$? **A**. $(12) = a$.

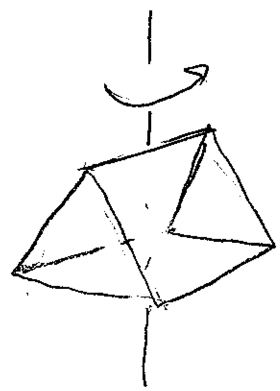
For any (G, S), the Cayley graph is very symmetric. In particular, G acts on Γ via
$$g \cdot v_h = v_{gh} \quad \text{(which doesn't break edges)}$$

In our example

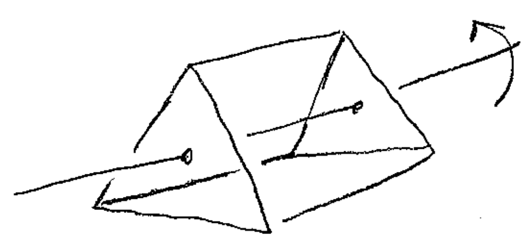
Comment on Expanders/ Geometric Group Theory!



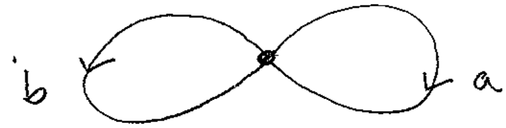
We have



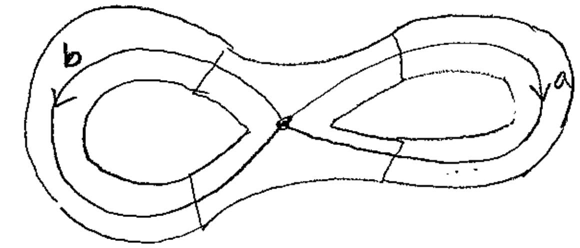
a rotates by π.



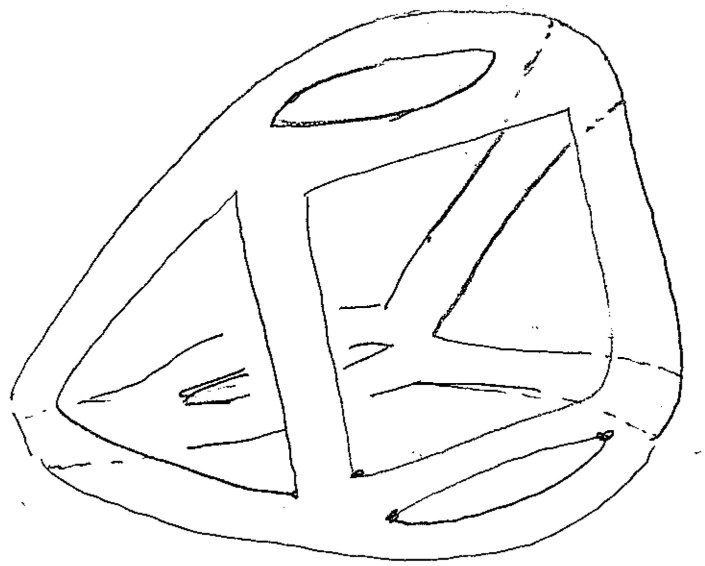
b rotates by $2\pi/3$.

What is Γ/G? A 

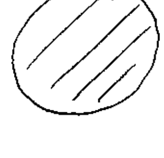
Now we want G to act on a surface, so

"thicken" Γ/G to 

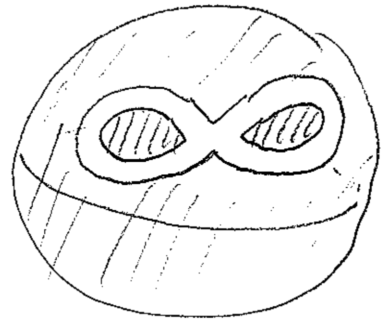
and corresponding Γ to



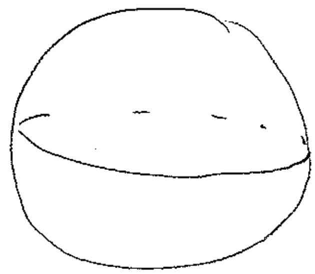
Now for each circle boundary component,

add a disc .

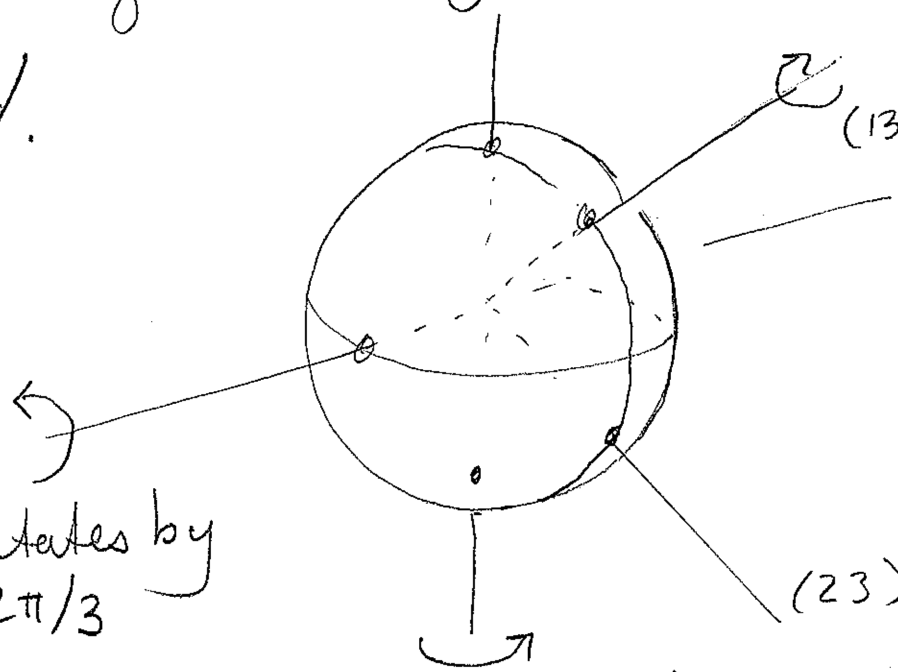
So Γ / G becomes

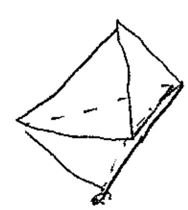
$X =$  $= \mathbb{P}^1_{\mathbb{C}}$

And G becomes

$Y =$  as well.

The action of G on Γ gives an action of G on Y.

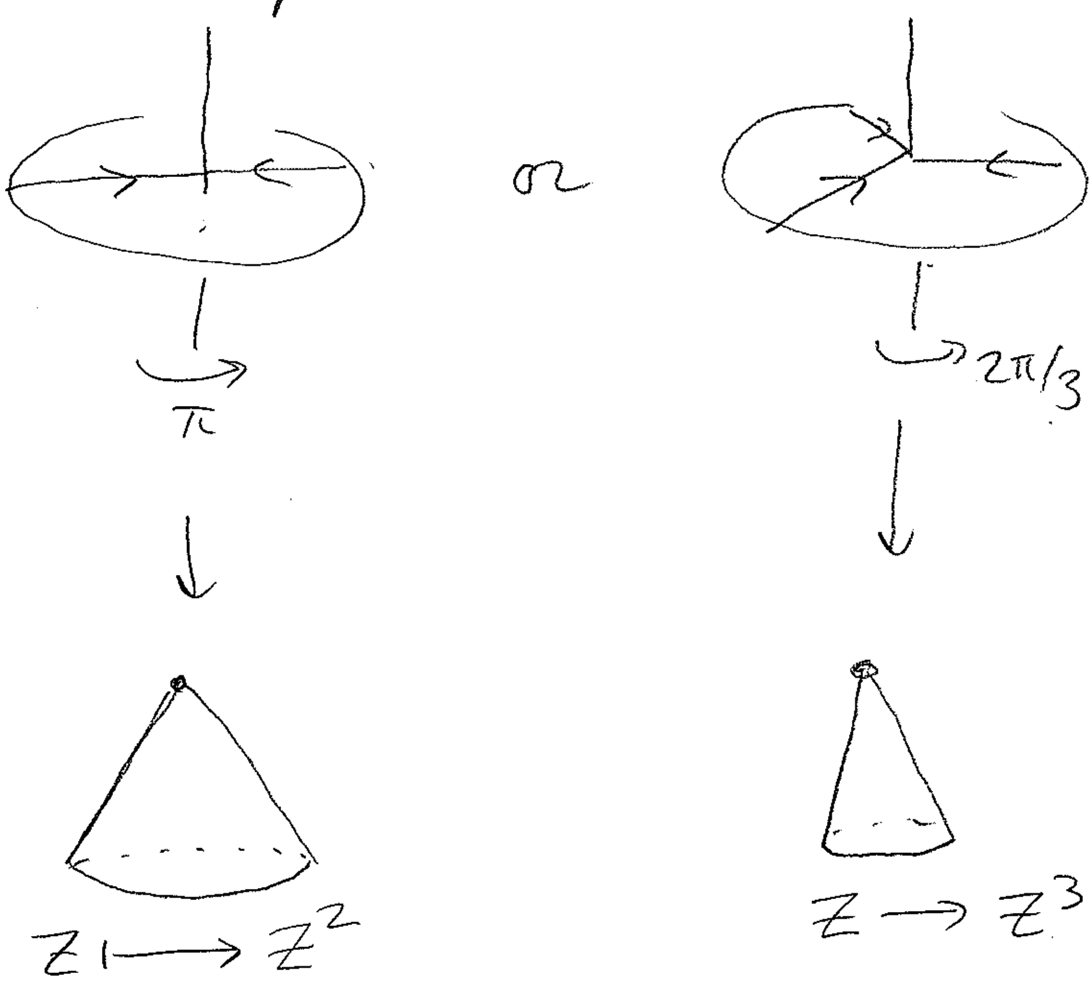


[S_3 = orientable isom of the bipyramid ]

What is $p: Y \to X$ like?

First, notice $\Gamma \to \Gamma/G$ is locally 1-1 (a homeomorphism). The same is true for $p: Y \to X$, except at the 8 points the are fixed by some elt of G (these are the centers of the added discs)

At these pts looks like



So, locally, p looks like a polynomial.

Now, we invoke the Riemann existence theorem to turn this into a honest rat'l map $\mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$. This will give an extension $K/\mathbb{C}(t)$ with Galois group S_3...

<u>Note</u>: The construction of $p: Y \to X$ from $\Gamma(G,S)$ is general. Its the R.E.T. that is hard...