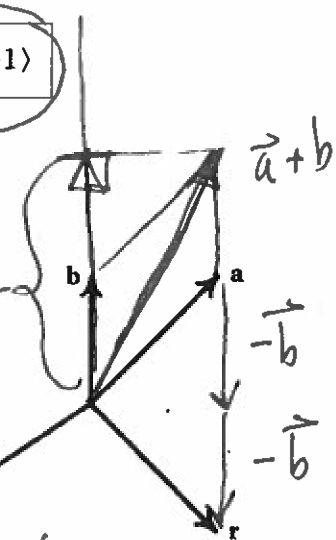
1. Circle the vector **n** that is normal to the plane containing the point $P = (1,2,2)$ and the line L parameterized by $x = 2$, $y = 1 + t$, and $z = 1 - t$. **(2 points)**

$\langle -2, 1, 1\rangle$	$\langle 0, 1, -1\rangle$	$\langle -2, 0, -1\rangle$	$\langle -2, -1, -1\rangle$

2. Consider the vectors **a**, **b**, **c** and **r** in the plane of this piece of paper. For each part, circle the best answer. **(1 point each)**



(a) $\mathbf{r} =$ | $\mathbf{a}-\mathbf{b}$ | $\mathbf{a}-2\mathbf{b}$ | $\mathbf{b}-\mathbf{c}$ | $2\mathbf{b}$ | $2\mathbf{a}$

(b) $\text{proj}_{\mathbf{b}}(\mathbf{a}+\mathbf{b}) =$ | $\mathbf{a}+\mathbf{b}$ | $\mathbf{a}$ | $\mathbf{b}$ | $2\mathbf{b}$ | $2\mathbf{a}$

(c) $\mathbf{b}\cdot\mathbf{c} =$ | positive | negative | zero

since $\measuredangle$ is $> \pi/2$.

3. Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = \begin{cases} \dfrac{xy+y^3}{x^2+y^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$ **(2 points each)**

(a) Check the box next to the only true statement below. The limit $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist because

- [ ] the numerator and denominator are both zero at $(0,0)$.
- [ ] the limit as one approaches $(0,0)$ along the lines $y=0$ and $x=0$ are different.
- [ ] the limit as one approaches $(0,0)$ along the the paths $y=x^2$ and $x=0$ are different.
- [x] the limit as one approaches $(0,0)$ along the lines $y=x$ and $x=0$ are different.

(b) Compute $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$; if a partial derivative does not exist, write "DNE".

$\dfrac{\partial f}{\partial x}(0,0) = \boxed{0}$ $\qquad$ $\dfrac{\partial f}{\partial y}(0,0) = \boxed{1}$

Scratch Space

1) $\vec{v}$ = vel. vector along L
$= \langle 0,1,-1\rangle$
$Q = \langle 2,1,1\rangle$ on L (@ $t=0$)
$\vec{w} = \overrightarrow{PQ} = \langle 1,-1,-1\rangle$
$\vec{n} = \vec{v}\times\vec{w} = \langle -2,-1,-1\rangle$

3a: $f(x,x) = \dfrac{x^2+x^3}{2x^2} = \dfrac{1}{2} + \dfrac{1}{2}x \to 1/2$ as $x\to 0$

$f(0,y) = y^3/y = y \to 0$ as $y \to 0$.

$f_x = \lim\limits_{h\to 0} \dfrac{f(h,0)-f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{0-0}{h} = 0$

$f_y = \lim\limits_{h\to 0} \dfrac{f(0,h)-f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{h}{h} = 1$

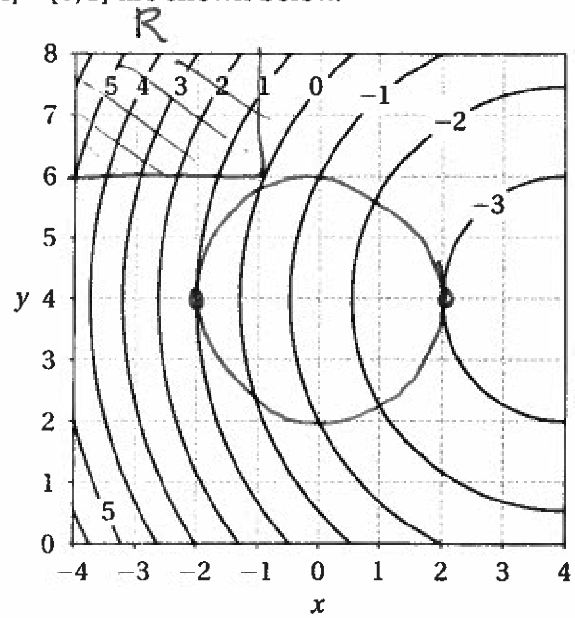
4. The level curves of a differentiable function $f(x, y)$ on $[-4, 4] \times [0, 8]$ are shown below.

(a) Circle the best estimate for $\int_{-4}^{-1} \int_{6}^{8} f(x, y)\, dy\, dx$.

−30	−24	−18	−12	−6	0	6	12	18	24	30

(2 points)

$$\iint_R f\, dA = \text{Area}(R) \cdot \text{Ave}(f \text{ on } R)$$
$$= 6 \cdot 3 = 18$$



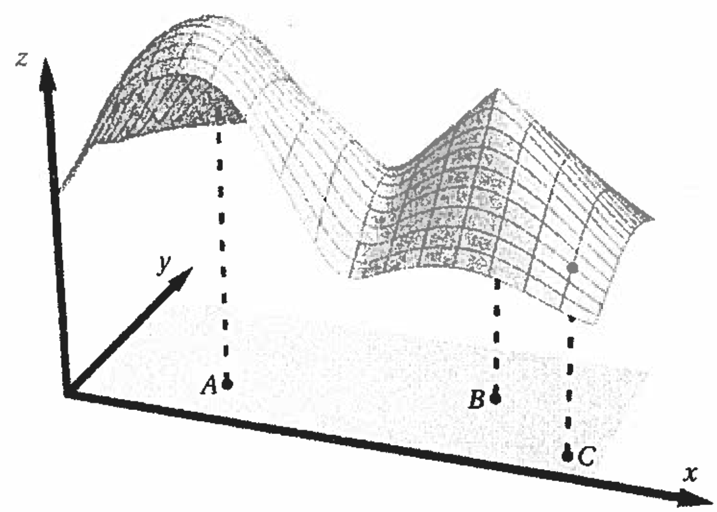
(b) Find the points on the curve $x^2 + (y-4)^2 = 4$ where f attains its absolute maximum and minimum values.

Max value = 1 at the point(s) (−2, 4) **(1 point)**

Min value = −3 at the point(s) (2, 4) **(1 point)**

(c) The absolute minimum value of f on the region $D = \{x^2 + (y-4)^2 \le 4\}$ is: −3 **(1 point)**

5. Consider the function $g\colon \mathbb{R}^2 \to \mathbb{R}$ whose graph is shown at right. Let A and B be the points in $\mathbb{R}^2$ corresponding to the two "peaks" of the graph, and C be the point in $\mathbb{R}^2$ corresponding to the dot on the graph. For each part, circle the answer that is most consistent with the picture. **(1 point each)**

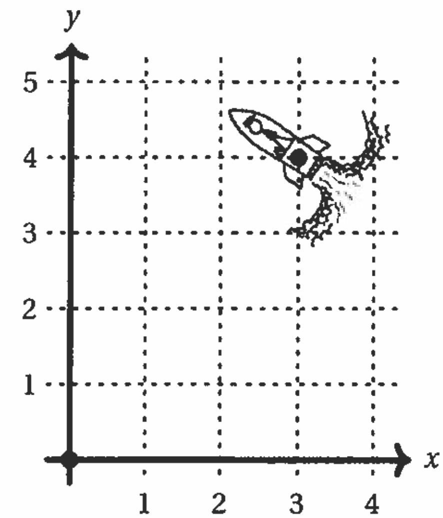


(a) At the point A, the function g is: continuous | differentiable | both | neither

(b) At the point B, the function g is: continuous | differentiable | both | neither

(c) At the point C, the function $\frac{\partial g}{\partial x}$ is: negative | zero | positive

6. An exceptionally tiny spaceship positioned as shown is traveling so that its x-coordinate *decreases* at a rate of 1/3 m/s and y-coordinate *increases* at a rate of 1/2 m/s. Use the Chain Rule to calculate the rate at which the distance between the spaceship and the point (0,0) is increasing. **(5 points)**



Distances in meters

Rocket courtesy of xkcd.com

$D = \sqrt{x^2+y^2}$ is $\sqrt{3^2+4^2} = 5$ at cur. pos.

$D_x = \frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2x = \frac{x}{D}$

$D_y = y/D$

$$\frac{dD}{dt} = \frac{\partial D}{\partial x}(3,4) \cdot \frac{dx}{dt} + \frac{\partial D}{\partial y}(3,4)\frac{dy}{dt}$$

$$= \frac{3}{5}\cdot\left(-1/3\right) + \frac{4}{5}\cdot\left(1/2\right)$$

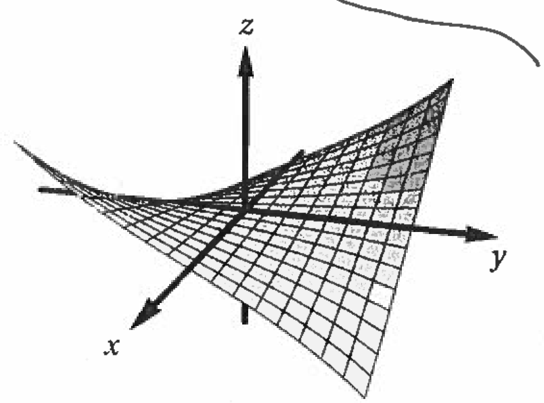
$$= -1/5 + 2/5 = 1/5$$

rate = $1/5$ m/s

In picture, graph contains x- and y-axes $\Rightarrow f_x(0,0) = f_y(0,0) = 0$.

$\Rightarrow$ as $f(0,0) = 0$ that tangent plane is $z = 0$

7. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the function whose graph is shown at right.



(a) Find the tangent plane to the graph at (0,0,0). **(1 point)**

Equation: $\boxed{0}\ x + \boxed{0}\ y + \boxed{1}\ z = \boxed{0}$

(b) The partial derivative $\dfrac{\partial^2 f}{\partial x \partial y}(0,0)$ is (circle your answer):

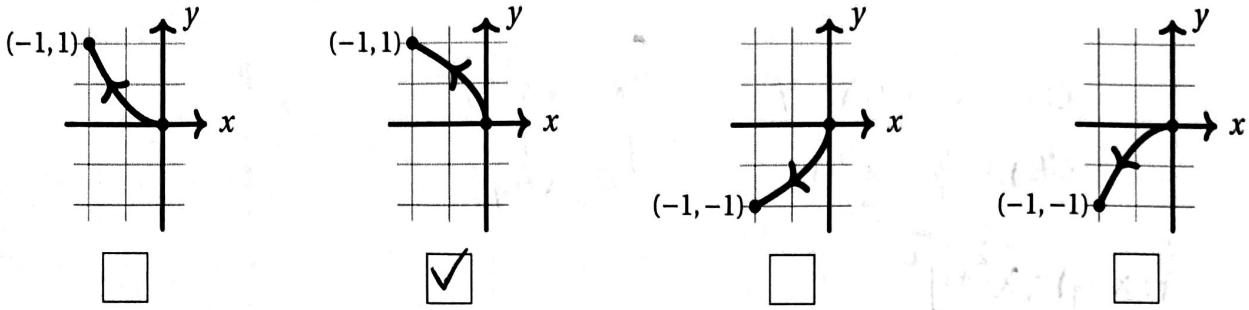
negative	zero	positive

(1 point)

8. Let C be the curve in $\mathbb{R}^2$ parameterized by $\mathbf{r}(t) = \langle -t^2, t\rangle$ for $0 \le t \le 1$.

$x = -y^2$

(a) Mark the picture of C from among the choices below. **(1 point)**



(b) For the vector field $\mathbf{F} = \langle y, -x\rangle$ compute $\int_C \mathbf{F}\cdot d\mathbf{r}$. **(3 points)**

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \langle t, t^2\rangle \cdot \langle -2t, 1\rangle \, dt = \int_0^1 -2t^2 + t^2 \, dt$$

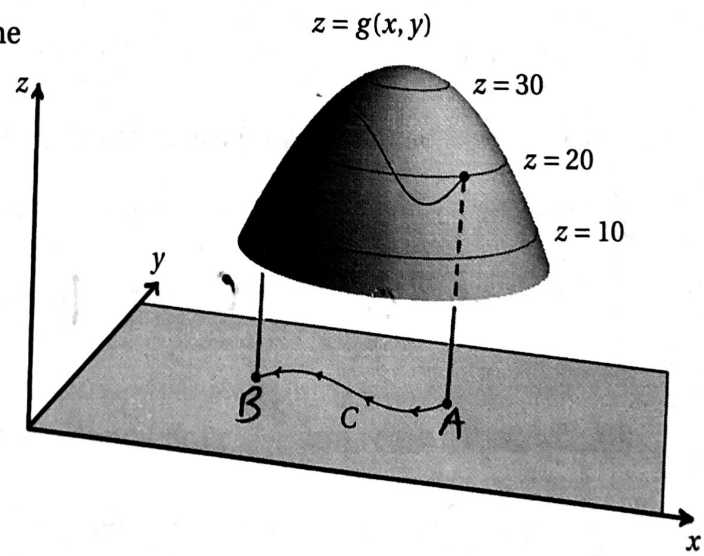
$$= -\int_0^1 t^2 \, dt = -\frac{t^3}{3}\Big|_0^1 = -\frac{1}{3}$$

$\int_C \mathbf{F}\cdot d\mathbf{r} = -\frac{1}{3}$

9. Let $g\colon \mathbb{R}^2 \to \mathbb{R}$ be the function whose graph is shown at right, and let C be the indicated curve in the xy-plane Evaluate the line integral:

$\int_C \nabla g \cdot d\mathbf{r} = \boxed{-10}$

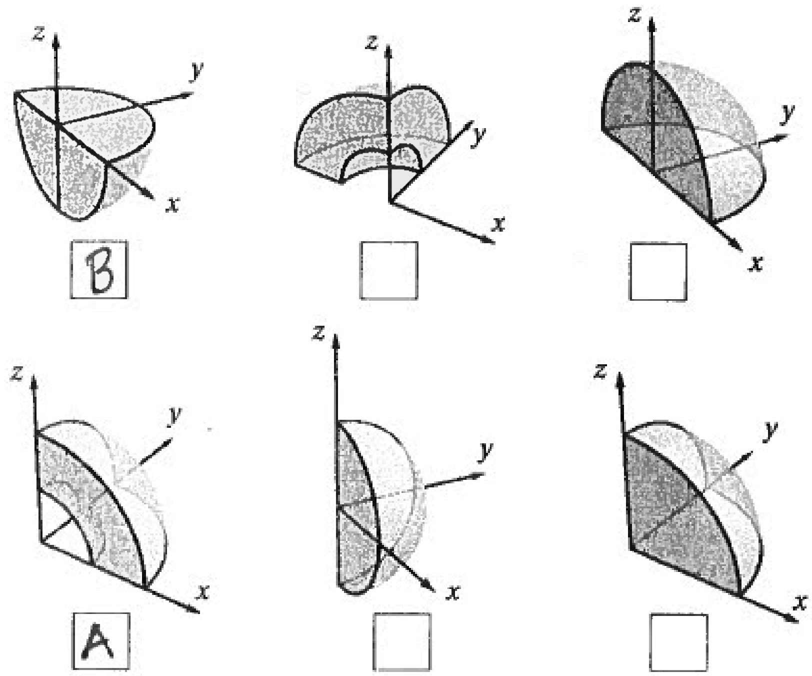
$$\int_C \nabla g\cdot d\vec{r} = g(B) - g(A)$$
$$= 10 - 20$$



10. For each of the given integrals, label the box below the picture of the corresponding region of integration in spherical coordinates. **(2 points each)**

$(A)\quad \displaystyle\int_0^{\pi/2}\int_0^{\pi/2}\int_1^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

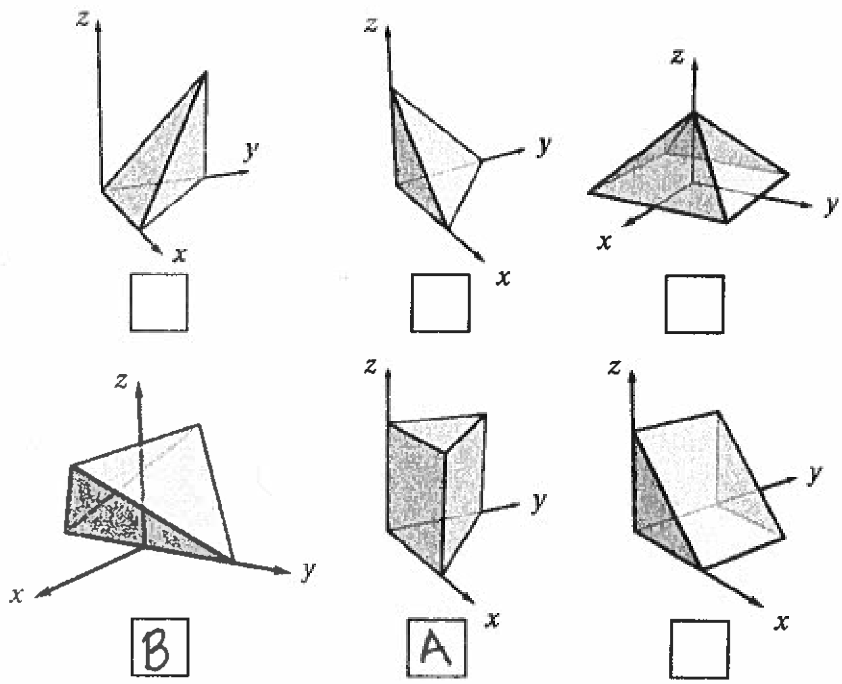
$(B)\quad \displaystyle\int_0^{\pi}\int_{\pi/2}^{\pi}\int_0^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$



11. For each of the given integrals, label the box below the picture of the corresponding region of integration.
(2 points each)

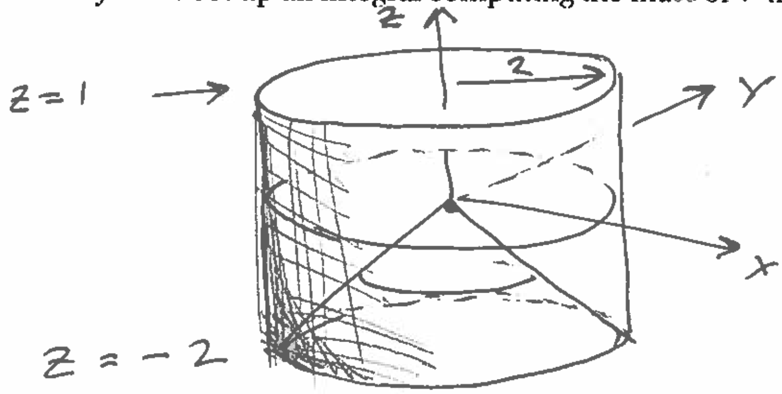
$(A)\quad \displaystyle\int_0^{1}\int_0^{1-x}\int_0^{1} f(x,y,z)\, dz\, dy\, dx$

$(B)\quad \displaystyle\int_0^{1}\int_{-1+z}^{1-z}\int_{-z}^{z} g(x,y,z)\, dx\, dy\, dz$



Scratch Space

12. Let V be the solid lying below the plane $z = 1$, above the surface $z = -\sqrt{x^2+y^2}$, and inside the cylinder $x^2+y^2 = 4$. Set up an integral computing the mass of V if the mass density is $\rho(x,y,z) = z+2$. **(4 points)**



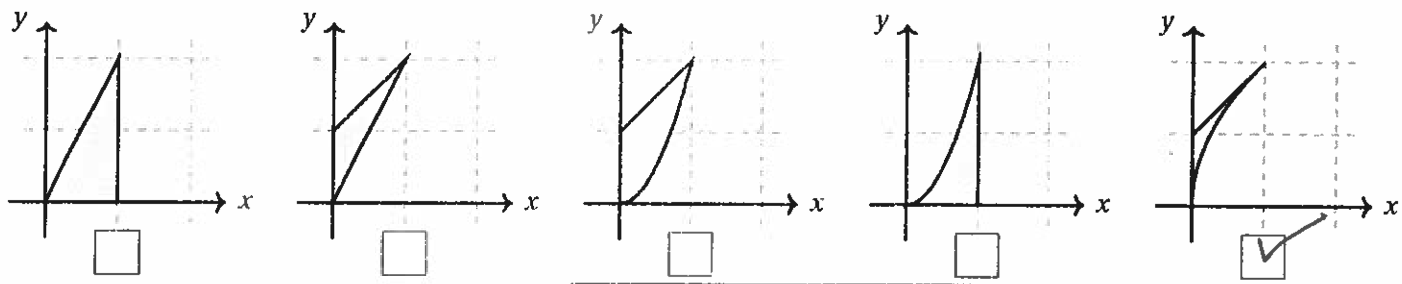
Use cylindrical coor

Be sure to fill in your variables of integration in the spaces after the d's.

Mass of $V = \displaystyle\int_0^{2\pi}\int_0^{2}\int_{-r}^{1} (z+2)\, r \; dz\, dr\, d\theta$

13. Consider the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(u,v) = (uv, u+v)$. Let D be the triangle in the uv-plane whose vertices are $(0,0), (1,0), (1,1)$. Let S be the region $T(D)$ in the xy-plane.

(a) Mark the box below the picture of S; here the dotted grids are made of unit-sized boxes. **(2 points)**

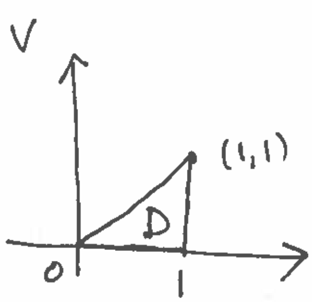


(b) Express $\iint_S y\, dA$ as an integral over D: $\displaystyle\int_0^1\int_0^u (u+v)(u-v)\; dv\, du$ **(2 points)**

Scratch Space

$J = \begin{pmatrix} v & u \\ 1 & 1 \end{pmatrix}$ $\det = v-u$ is < 0 as $u \geq v$ on D

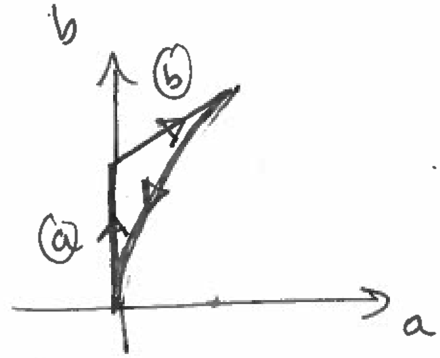
So $|J| = u - v$



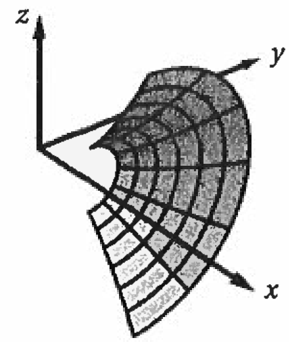
a) $T(u,0) = (0,u)$

b) $T(1,v) = (v, v+1)$

c) $T(u,u) = (u^2, 2u)$



14. The curve $y = x^2$ in the xy-plane is revolved about the x-axis in $\mathbb{R}^3$ to produce a surface. Parameterize the **portion of this surface with $y \geq 0$ and $1/2 \leq x \leq 1$** which is shown at right. Be sure to specify the domain D. **(3 points)**



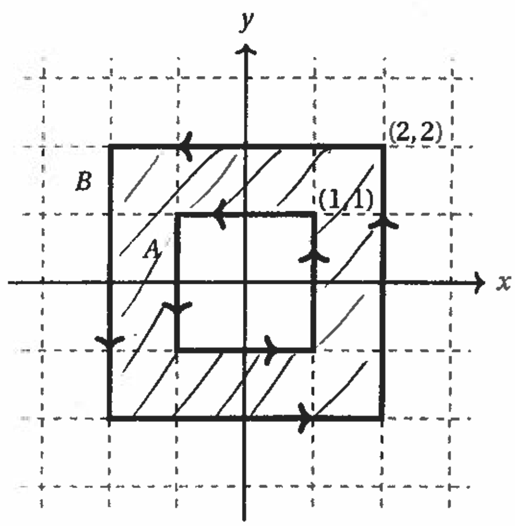
Use rotated cylindrical coordinates.

On this surface $r = x^2$.

$\mathbf{r}(u,v) = \langle\ u\ ,\ u^2 \cos v\ ,\ u^2 \sin v\ \rangle$

$D = \{\ (u,v)\ |\ 1/2 \leq u \leq 1\ ,\ -\pi/2 \leq v \leq \pi/2\ \}$

15. A vector field $\mathbf{F} = \langle P, Q\rangle$ is defined on the plane minus the origin and $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} + 2$ for all $(x,y) \neq (0,0)$. Let A and B be the two *oriented* curves shown at the right drawn against a grid of unit squares, and suppose $\int_B \mathbf{F}\cdot d\mathbf{r} = 16$. Evaluate the integral:



$\int_A \mathbf{F}\cdot d\mathbf{r} =$ | −24 | −16 | −8 | 0 | 8 | 16 | 24 |

(2 points)

Let D be the region shown. Then

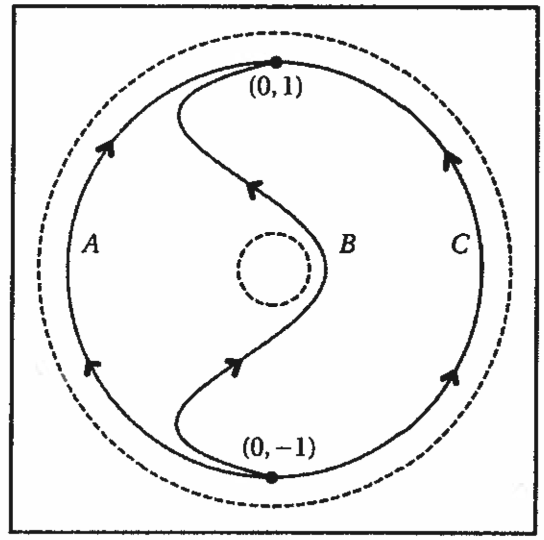
$\int_B \vec{F}\cdot d\vec{r} - \int_A \vec{F}\cdot d\vec{r} = \int_{\partial D} \vec{F}\cdot d\vec{r}$

Scratch Space

$= \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = 2\ \text{Area}(D) = 24$

So $\int_A \vec{F}\cdot d\vec{r} = \int_B \vec{F}\cdot d\vec{r} - 24 = -8$.

16. The region D defined by $\{0.03 < x^2 + y^2 < 1.3\}$ is shown at right. The curves A, B, C are within this region. Each curve starts at $(0,-1)$ and ends at $(0,1)$. Suppose that $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ is a differentiable vector field defined on D with the properties



$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \qquad \int_A \mathbf{F}\cdot d\mathbf{r} = 2, \quad \text{and} \quad \int_C \mathbf{F}\cdot d\mathbf{r} = -1.$$

(a) The region D is simply connected. **(1 point)**

True	False

(b) $\mathbf{F}$ is conservative. **(1 point)**

Yes	No	Cannot determine

(c) Find $\int_B \mathbf{F}\cdot d\mathbf{r}$. **(1 point)**

−3	−2.5	−2	−1.5	−1	−0.5	0	0.5	1	1.5	2	2.5	3

17. For this problem, $\mathbf{G} = \langle yz + 2x^2, 2xy, xy^2 \rangle$ and S is the boundary of the cube $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$ oriented with the outward pointing normal vector $\mathbf{n}$. Circle the best response for each of the following.

(a) $\iint_S \mathbf{G}\cdot\mathbf{n}\, dS =$

−5	−3	−1	0	1	3	5

(2 points)

(b) $\iint_S (\operatorname{curl}\mathbf{G})\cdot\mathbf{n}\, dS$ is

negative	zero	positive

(1 point)

(c) Suppose a charge Q is placed at $\mathbf{p} = \langle 1/2, 1/2, 1/2\rangle$ and let $\mathbf{E} = \dfrac{Q}{4\pi\epsilon_0}\dfrac{1}{|\mathbf{r}-\mathbf{p}|^3}(\mathbf{r}-\mathbf{p})$ for $\mathbf{r} = \langle x,y,z\rangle$ be the resulting electric field. Then $\iint_S \mathbf{E}\cdot\mathbf{n}\, dS = \boxed{Q/\varepsilon_0}$ **(1 point)**

Scratch Space

a) $\text{Flux} = \iiint_{\text{Cube}} \operatorname{div}\vec{G}\, dV = \int_0^1\int_0^1\int_0^1 4x + 2x + 0 \, dx\, dy\, dz$

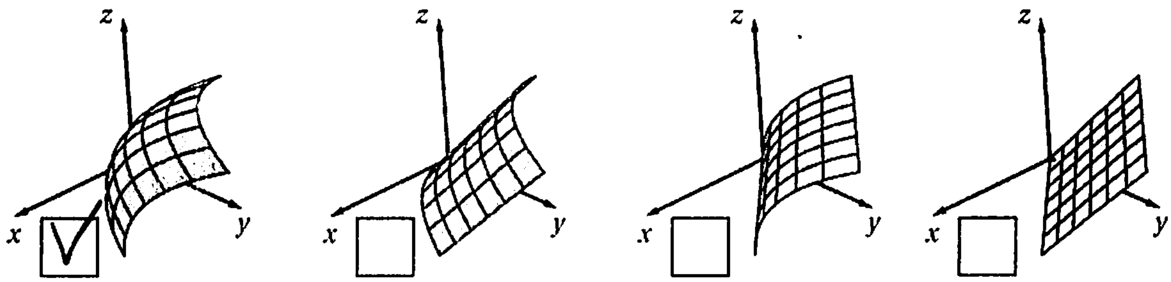
$= \int_0^1 6x\, dx = 3x^2\Big|_0^1 = 3$

b) Since S is closed (no boundary), $\iint_S (\operatorname{curl}\vec{G})\cdot\vec{n}\, dS = 0$.

c) Gauss's Law!

18. Consider the surface S parameterized by $\mathbf{r}(u,v) = \langle u,\ u^2+v^2,\ v\rangle$ defined on $D = \{(u,v) \mid 0 \le u \le 1, 0 \le v \le 1\}$ and oriented by the normal vector $\mathbf{n}$ with positive second component.

(a) Mark the box below the picture of S. **(2 points)**



(b) Compute $\iint_S \langle z, 3, -x\rangle \cdot \mathbf{n}\, dS$. **(4 points)**

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2u & 0 \\ 0 & 2v & 1 \end{vmatrix} = \langle 2u, -1, 2v\rangle$$

note wrong direction, fixed here,

$$\iint_S \langle z, 3, -x\rangle \cdot \vec{n} = \int_0^1\int_0^1 \langle v, 3, -u\rangle \cdot (-\vec{r}_u \times \vec{r}_v)\, du\,dv$$

$$= \int_0^1\int_0^1 -2uv + 3 + 2uv\ du\,dv$$

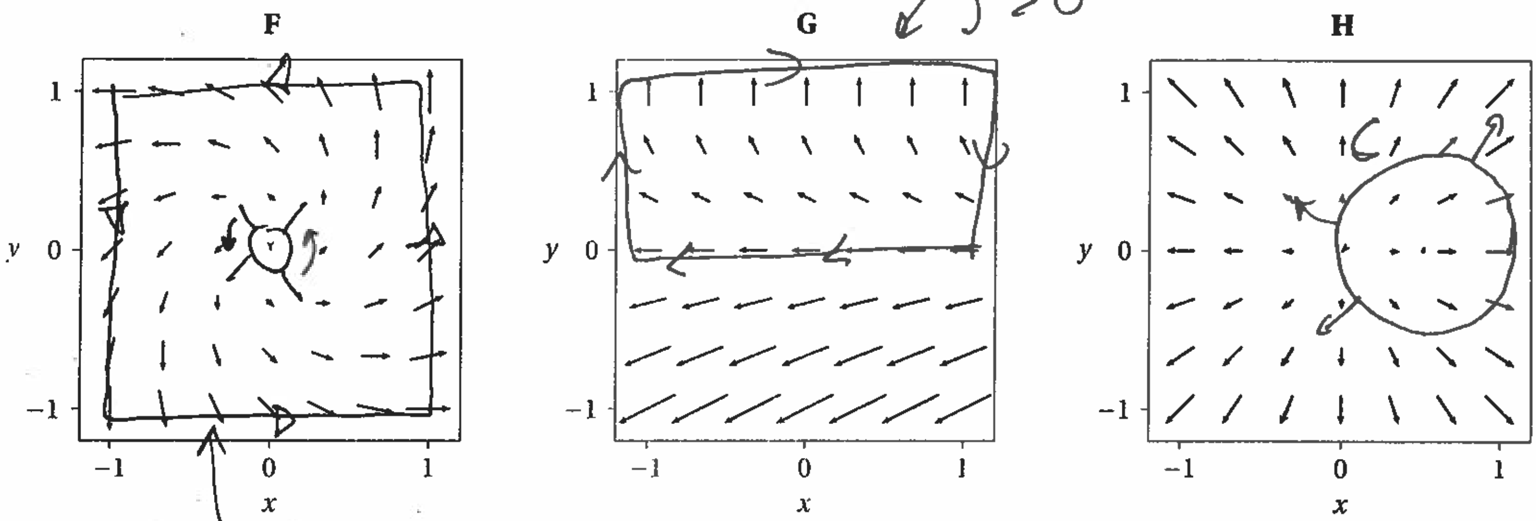
$$= \int_0^1\int_0^1 3\, du\,dv = 3$$

$\iint_S \langle z, 3, -x\rangle \cdot \mathbf{n}\, dS =$ $\boxed{3}$

(c) Fill in the integrand so that the surface area of S is: $\boxed{\int_0^1\int_0^1 \sqrt{1+4(u^2+v^2)}\ du\,dv}$ **(1 point)**

Scratch Space

19. Three vector fields are shown below, **exactly one** of which is conservative. For each of the following questions, circle the best answer.



$\int > 0$

(a) The conservative vector field is: | F G H | **(2 points)**

(b) The vector field $\langle y-1,\ y\rangle$ is: | F G H | **(1 point)**

(c) The function div **H** is constant. The value of div **H** at any point is:

| negative zero positive | **(1 point)**

(d) The vector field curl **F** is constant. The value of curl **F** at any point is:

| $\langle 0, 0, -1\rangle$ $\langle 0, 0, 0\rangle$ $\langle 0, 0, 1\rangle$ | **(1 point)**

(e) Let C be the circle $\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{4}$, and **n** the outward pointing normal vector in the plane.

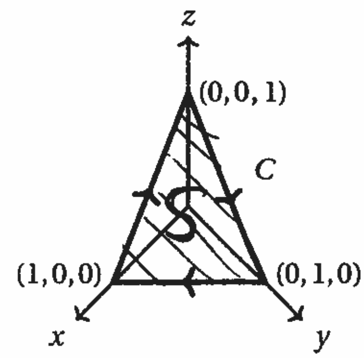
The 2D flux $\int_C \mathbf{H}\cdot\mathbf{n}\,ds$ is: | negative zero positive | **(1 point)**

$\parallel$

Scratch Space

$\iint \operatorname{div} \vec{H}\, dA > 0$

20. For this problem, $\mathbf{F} = \langle x^2 - 2y,\ y^2 - 2z,\ z^2 - 2x\rangle$ and C is the oriented closed curve made from three straight line segments shown at the right.



(a) Compute $\operatorname{curl}\mathbf{F}$. **(2 points)**

$$\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2-2y & y^2-2z & z^2-2x \end{vmatrix} = \langle 2,2,2\rangle$$

$\operatorname{curl}\mathbf{F} = \boxed{\langle 2, 2, 2\rangle}$

(b) Compute $\operatorname{div}\mathbf{F}$. **(1 point)**

$$\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 2x + 2y + 2z$$

$\operatorname{div}\mathbf{F} = \boxed{2(x+y+z)}$

(c) Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$. **(4 points)** Use Stokes' with S = triangle oriented towards $\vec{0}$. Param by $\vec{r}(u,v) = \langle u, v, 1-u-v\rangle$ With D =

Then $\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \langle 1,1,1\rangle$

which points the wrong way. So use $-\vec{r}_u \times \vec{r}_v$

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\operatorname{curl}\vec{F})\cdot\vec{n}\,dA = \iint_D \underbrace{\langle 2,2,2\rangle}_{\operatorname{curl}\vec{F}} \cdot \langle -1,-1,-1\rangle\,du\,dv$$

$$= -6\iint_D 1\,du\,dv = -6\,\text{Area}(D) = -3$$

$\int_C \mathbf{F}\cdot d\mathbf{r} = \boxed{-3}$

(d) $\int_C \operatorname{div}\mathbf{F}\,ds$ is: negative zero positive **(1 point)**

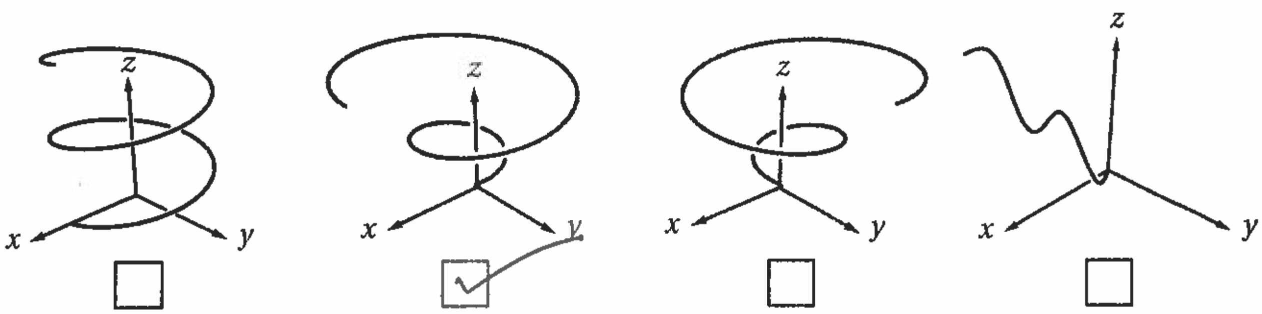
Since $\operatorname{div} F$ is ≥ 0 on C.

(e) Is $\mathbf{F}$ conservative? yes no **(1 point)**

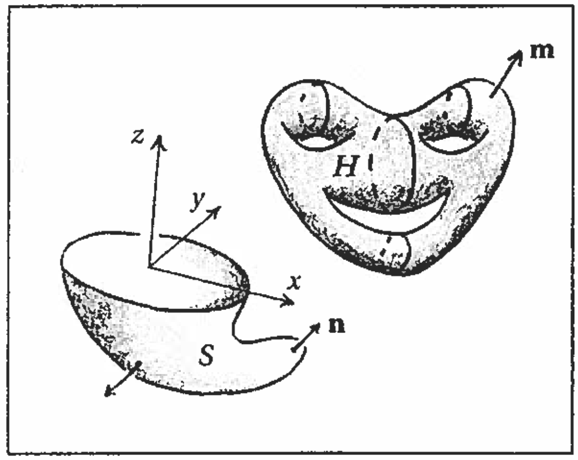
by (b).

$r(4\pi) = \langle 4\pi, 0, 4\pi \rangle$

21. Mark the picture of the curve in $\mathbb{R}^3$ parameterized by $\mathbf{r}(t) = \langle t\cos t,\ t\sin t,\ t\rangle$ for $0 \le t \le 4\pi$. **(2 points)**



22. Let S and H be the surfaces at right; the boundary of S is the unit circle in the xy-plane, and H has no boundary. **1 pt each**



(a) The integral $\iint_H x^2y^2 + z^2\,dS$ is:

negative zero positive

Since integrand is pos. on H

(b) The vector field $\mathbf{F} = \langle y+z,\ -x,\ yz\rangle$ has $\operatorname{curl}\mathbf{F} = \langle z,\ 1,\ -2\rangle$.

The flux $\iint_S (\operatorname{curl}\mathbf{F})\cdot\mathbf{n}\,dS$ is:

-5π -4π -3π -2π $-\pi$ 0 π 2π 3π 4π 5π

(c) For $\mathbf{G} = \langle x,\ y,\ z\rangle$, the flux $\iint_S \mathbf{G}\cdot\mathbf{n}\,dS$ is:

negative zero positive

(d) For $\mathbf{E} = \langle z,\ x,\ 2\rangle$, the flux $\iint_S \mathbf{E}\cdot\mathbf{n}\,dS$ is:

-5π -4π -3π -2π $-\pi$ 0 π 2π 3π 4π 5π

Scratch Space

b) Use Stokes' with

C oriented clockwise: $\vec{r} = (\sin t, \cos t, 0)$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \langle \cos t, -\sin t, 0\rangle \cdot \langle \cos t, -\sin t, 0\rangle\, dt$$

$$= \int_0^{2\pi} 1\, dt = 2\pi$$

c) Let D be the disc in the xy-plane bounded by C, R the region with $\partial R = S + D$. Then $\iiint_R \underbrace{\operatorname{div} G}_{=3}\, dV$

$$= \underbrace{\iint_D \vec{G}\cdot d\vec{S}}_{} + \iint_S \vec{G}\cdot dS$$

$= 0$ as $\vec{n} = (0,0,1)$ and $\vec{G} = (x, y, 0)$

d) Similar, but $\operatorname{div}\vec{E} = 0$

$$\Rightarrow \iint_S \vec{E}\cdot\vec{n}\, dA = \iint_D \vec{E}\cdot(0,0,-1)\, dA$$

$$= \iint_D -2\, dA = -2\pi.$$