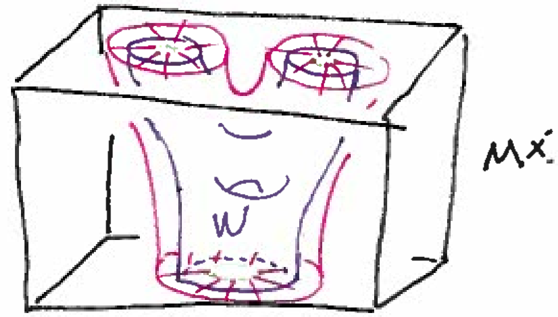
Framed bordism and homotopy groups of spheres.

A framing of a [smooth embed...] submanifold $V^{k-n} \subseteq M^k$ is an embedding $\phi : V \times \mathbb{R}^n \longrightarrow M^k$ where $\phi(v,0) = v$ for all $v \in V$.

$$\Omega^{fr}_{k-n,M} = \left\{ \begin{matrix} \text{framed submanifolds} \\ \text{modulo framed bordism} \end{matrix} \right\}$$

M ← closed



Thm: The collaspe map $\Omega^{fr}_{k-n,M} \xrightarrow{c} [M, S^n]$ is a bijection

$$(V,\phi) \longmapsto \begin{cases} M \setminus \phi(V\times\mathbb{R}^n) \longmapsto \infty \\ \pi_{\mathbb{R}^n} \circ \phi^{-1} \text{ on } \phi(V \times \mathbb{R}^n) \end{cases}$$

For details, see Davis-Kirk, Chapter 8.

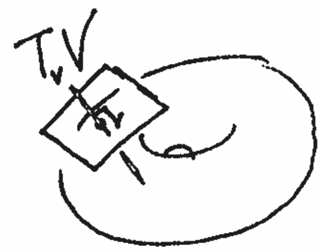
Take $M = S^k$, so that $\Omega^{fr}_{k-n,S^k} \cong [S^k, S^n]$

$= \pi_k S^n = \pi^S_{k-n}$ when $n \geqslant k-n+2$.

π^S_{k-n} ← stable $(k-n)$-stem

Cor. For $k \geqslant 2l+2$, $\Omega^{fr}_{l,S^k} \cong \pi^S_l = \pi_k S^{k-l}$

$V^\ell \subseteq (S^k, g_{\text{round}})$



$$\begin{array}{ll} TV & \text{a fiber bundle} \\ \downarrow & \text{with fiber } \mathbb{R}^\ell \\ V & \text{and structure gp } GL_n(\mathbb{R}) \text{ or } O(n). \end{array}$$

$$\begin{array}{ll} NV & N_vV = \{ w \in T_vS^n \mid w \perp \text{ to } T_vV \} \\ \downarrow & \quad [\text{or } N_vV = T_vS^n / T_vV] \\ V & \end{array}$$

vector bundle with fibers $\cong \mathbb{R}^{k-\ell}$

<u>Prop</u>: A framing of $V \subseteq S^k$ is equivalent to a trivialization $NV \cong V \times \mathbb{R}^{k-\ell}$. That is a collection $X_1, \ldots, X_{k-\ell}$ of sections of NV which give a basis for each N_vV.

Now inclusion $S^k \hookrightarrow S^{k+1}$ induces

$\Omega^{fr}_{\ell, S^k} \longrightarrow \Omega^{fr}_{\ell, S^{k+1}}$ [where the new trivialization of $N(V \subseteq S^{k+1})$ comes by adding a vector field given by the fact that the normal bundle of S^k in S^{k+1} is trivial]

Moreover

$$\begin{tikzcd} \Omega^{fr}_{\ell, S^k} \arrow[r, "\iota"] \arrow[d, "e"'] & \Omega^{fr}_{\ell, S^{k+1}} \arrow[d, "c"] \\ {[S^k, S^{k-\ell}]} \arrow[r, "\text{Suspension}"'] & {[S^{k+1}, S^{k-\ell+1}]} \end{tikzcd}$$

since

$$N(V \subseteq S^{k+1}) = N(V \subseteq S^k) \oplus \varepsilon^1$$

where here ε^j denotes the <u>trivialized</u> $\mathbb{R}^j$ bundle over V.

<u>Lemma</u>: $V^\ell \subseteq S^k$ a closed submanifold of S^k.

(a) A normal framing $NV \underset{\gamma}{\xrightarrow{\cong}} \varepsilon^{k-\ell}$ induces a trivialization

$$\bar{\gamma} : TV \oplus \varepsilon^{k-\ell+1} \underset{\bar{\gamma}}{\longrightarrow} \varepsilon^{k+1}$$

(b) A trivialization of $TV \oplus \varepsilon \longrightarrow \varepsilon^{\ell+1}$ induces a trivialization $NV \oplus \varepsilon^{\ell+1} \cong \varepsilon^{k+1}$

<u>Point</u>: If $TS^k \cong \varepsilon^k$ then a trivialization of TV induces one of NV and vice versa. This isn't the case, but $TS^k \oplus \varepsilon \cong \varepsilon^{k+1}$ since $TS^k \oplus \varepsilon \subseteq T\mathbb{R}^{n+1}$ which is trivial.

Def: A stable framing of an ℓ-manifold V is an equivalence class of trivializations $TV \oplus \varepsilon^n$

where $TV \oplus \varepsilon^{n_i} \xrightarrow[t_i]{\cong} \varepsilon^{\ell + n_i}$ are equivalent if $\exists N > n_1, n_2$

where

$$t_i \oplus id : TV \oplus \varepsilon^{n_i} \oplus \varepsilon^{N - n_i} \longrightarrow \varepsilon^{\ell + N}$$

are homotopic.

$$\Omega^S_\ell = \left\{ \begin{array}{l} \text{stably framed closed} \\ \ell\text{-manifolds} \end{array} \right\} \Big/ \begin{array}{l} \text{stably framed} \\ \text{bordism.} \end{array}$$

Thm: For large k, $\Omega^S_\ell \cong \Omega^{fr}_{\ell, S^k}$.

Map: Any $V \hookrightarrow S^{2\ell}$. Including $S^{2\ell}$ in some large S^k lets us turn the stable framing on V into a normal framing on $V \subseteq S^k$.

Onto is clear: Any elt of Ω^{fr}_{ℓ, S^k} gives a stable framing of the assoc. submfld V.

Finally 1-1 is because any two embeddings of $V \hookrightarrow S^k$ are isotopic in S^{k+2} if $k \geq 2\ell$.
(in particular cobordant).

<u>Thm.</u> Ω^s_ℓ, the set of bordism classes of stably framed ℓ-manifolds (closed smooth) is $\cong \pi^s_\ell$.