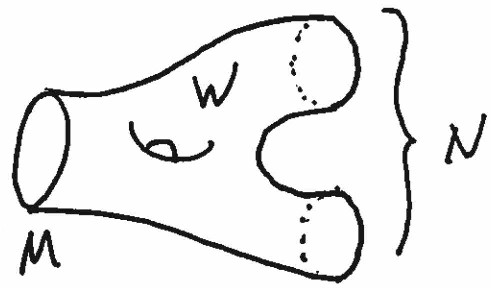
Bordism I: Unoriented cobordism ring

Suppose M and N are smooth closed (cpt w/o bdy) n-manifolds. Say that M and N are (co)bordant if there is a smooth cpt $(n+1)$-manifold W with

Diffeom. to $\partial W = M \amalg N$

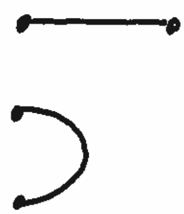


Define

$$\Omega_n = \left\{ \begin{array}{l} \overbrace{\text{equivalence classes}}^{\text{explain}} \\ \text{of smooth closed} \\ n\text{-manifolds, up to ~~cop~~ cobordism.} \end{array} \right\}$$

Ex:

$\Omega_0 = \{[\phi], [pt]\}$



so $[\cdot\cdot\cdot] = [\cdot]$

$\Omega_1 = \{[\phi]\}$

$\Omega_2 = \{[\phi], [\mathbb{R}P^2]\}$

$\Omega_3 = \{[\phi]\}$

$\Omega_4 = \{[\phi], [\mathbb{R}P^4]\}$ and $[\mathbb{R}P^2 \times \mathbb{R}P^2]$

Algebra structure: $\Omega_* = \bigcup \Omega_n$

$+$ disjoint union [respects equivalence relation!]

$\times$ product of manifolds $\Omega_n \times \Omega_m \longrightarrow \Omega_{n+m}$

Additive identity: $[\emptyset]$ (have identified all of them...) ②

Mult. identity: $[pt]$

Characteristic is two: $\partial(M\times I) = M\times\{0\} \amalg M\times\{1\}$

and so $[M]+[M] = [M \amalg M] = 0.$

This gives ~~[illegible]~~ that $-[M] = [M]$ so we have additive inverses and Ω_* is a ring, in fact an algebra over $\mathbb{F}_2$.

Theorem (Thom) Ω_* is a polynomial algebra over $\mathbb{F}_2$ on generators u_i for $i>1$ and $i \neq 2^r-1$ for some $r\in\mathbb{N}$.

Explicit generators: $u_{2i} = [\mathbb{RP}^{2i}]$

For ~~[illegible]~~ $m<n$ define $H_{n,m}$ to be the hypersurface in

$\mathbb{RP}^n \times \mathbb{RP}^m$ cut out by $x_0y_0 + \cdots + x_my_m = 0$.

$[x_i]$ $[y_i]$

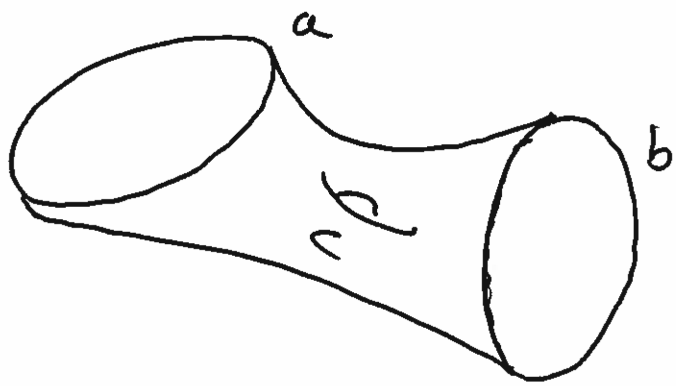
If i is odd and not 2^r-1, can write $i = 2^p(2q+1)-1$

for $p,q\geq 1$. Take $u_i = [H_{2^{p+1}q,\,2^p}]$

Bordism is a very coarse equivalence. Why do we consider it? Recall

$$H_k(X) = \frac{k\text{-dim'l things w/o boundary}}{\text{boundaries of } (k+1)\text{-dim'l things}}$$

Geometric intuition: X a manifold, cycles are submanifolds, boundaries are submflds with boundary.



$\partial c = a + b$ ($\mathbb{F}_2$-coeffs)

For any space X we can define:

$$C_n^{\mathrm{Bord}}(X) = \left\{ f : M \to X \;\middle|\; \begin{array}{l} M \text{ a compact smooth } n\text{-mfld} \\ f \text{ a cont map} \end{array} \right\}$$

which has an addition-like operation:

$$(f_1 : M_1 \to X) + (f_2 : M_2 \to X) = (f_1 \amalg f_2 : M_1 \amalg M_2 \to X)$$

and a boundary map
$$C_n^{\mathrm{Bord}}(X) \longrightarrow C_{n-1}^{\mathrm{Bord}}(X)$$
$$(f : M \to X) \longmapsto f|_{\partial M}$$

Note that $\partial^2 = 0$! So form homology $H_*^{\mathrm{Bord}}(X)$.

<u>Cycles</u>: $f: M \to X$ with M closed.

<u>Char</u>: $[f] + [f] = 0$ since $f + f = \partial(M \times I \xrightarrow[f \circ \pi_M]{} X)$

so $H_*^{Bord}(X)$ is a vector space $/ \mathbb{F}_2$.

<u>Homotopy</u>: $f_{1,2}: M \to X$ (clsd) are homotopic then $[f_1] = [f_2]$

<u>Functorial</u>: $F: X \to Y$ gives $F_*: H_*^{Bord}(X) \to H_*^{Bord}(Y)$

In fact H_*^{Bord} is a homology theory of CW complexes, but its not $H_*(\cdot\,; \mathbb{F}_2)$ since

$$H_*(pt; \mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & \text{for } * = 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$H_*^{Bord}(pt) \cong \Omega_* .$$

However, do have a map $H_*^{Bord}(X) \to H_*(X; \mathbb{F}_2)$

$$[f: M \to X] \longmapsto f_*[M]$$

[Also an oriented version, etc.]

Thom computed Ω_* in 1954. He did so by identifying it with $\pi_*(TO)$. [Emphasize theme.]

prespectra.

$Gr_n(\mathbb{R}^k)$ = Grassmannian of n planes in $\mathbb{R}^k$ where $n<k$.

($n=1$ is $\mathbb{R}P^{k-1}$)

There is a nat'l vector bundle $E_{n,k} \to Gr_n(\mathbb{R}^k)$

$$E_{n,k} = \{ Gr_n(\mathbb{R}^k) \times \mathbb{R}^k \text{ of form } (P_n, \text{pt in } P_n) \}$$

Consider

$Gr_n(\mathbb{R}^\infty) = \bigcup_{k>n} Gr_n(\mathbb{R}^k) =$ n-planes in $\mathbb{R}^\infty$, since any $x \in \mathbb{R}^\infty$ has only finitely many nonzero x_i

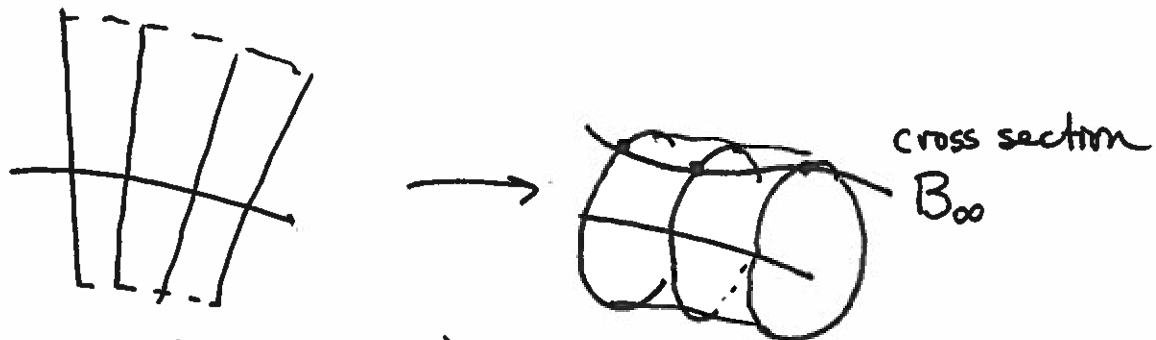
which has ~~[illegible]~~ a rank n vector bundle $E_n \to Gr_n(\mathbb{R}^\infty)$. This is the <u>universal rank n bundle</u>: if X is any paracompact space, then

Also called $BO(n)$

$$\left\{ \begin{matrix} \text{Isom classes} \\ \text{of } \mathbb{R}^n \text{ vector} \\ \text{bundles over } X \end{matrix} \right\} = [X, Gr_n(\mathbb{R}^\infty)]$$

Thom space: $E \to B$ vector bundle with fiber $\mathbb{R}^n$.

Let $Sph(E)$ be the bundle where each fiber is S^n as the one-pt compactification of the copy of $\mathbb{R}^n$ at that pt.



Define $T(E) = Sph(E)/B_\infty$. If B is cpt, this is just the 1-pt compactification of E.

Thom isomorphism: $\exists\, u \in \tilde{H}^n(T(E))$ so that

$$\Phi : H^*(B) \longrightarrow \tilde{H}^{*+n}(T(E))$$
$$x \longmapsto x \cup u$$

suitablely interpreted.

is an isomorphism.

Aside for now.

Define $TO(n) = T(E_n \to Gr_n(\mathbb{R}^\infty))$. Then

$$\Omega_k = \pi_{n+k}(TO(n)) \text{ for all large } n.$$

This is a sensible thing to do since have

$$Gr_n(\mathbb{R}^\infty) \longrightarrow Gr_{n+1}(\mathbb{R}^\infty)$$

where pull-back bundle is $\underbrace{E_n \oplus \begin{pmatrix}\text{trivial} \\ \mathbb{R}\text{-bundle}\end{pmatrix}}$. Thom space is functorial, and $T(\quad)$ is $\Sigma\, TO(n)$

Get maps $\Sigma\, TO(n) \xrightarrow{q_n} TO(n+1)$ giving

$$\begin{tikzcd} \pi_{n+k}(TO(n)) \arrow[r, "\Sigma"] \arrow[rr, bend right] & \pi_{n+k+1}(\Sigma\, TO(n)) \arrow[r, "q_{n*}"] & \pi_{n+k+1}(TO(n+1)) \end{tikzcd}$$

turns out composition is $\cong$ for n large and so may define

$$\pi_k(TO) = \varinjlim_n \pi_{n+k}(TO(n)).$$