Lecture 32: Cohomology via $K(G,n)$'s

"pointed spaces"
X, Y spaces with basepts. Define $\langle X, Y\rangle = \left\{\begin{matrix}\text{base pt}\\ \text{pres maps}\end{matrix}\right\} \Big/ \begin{matrix}\text{base pt}\\ \text{pres.}\\ \text{homotopy}\end{matrix}$

Cf. $[X, Y] = \left\{\begin{matrix}\text{homotopy}\\ \text{classes of maps}\end{matrix}\right\}$

Ex: $\pi_n X = \langle S^n, X\rangle$

Thm: X a CW complex, G an abelian gp, $n > 0$. There exists a natural bijection $T: \langle X, K(G,n)\rangle \to H^n(X; G)$. This has the form $T([f]) = f^*\alpha$ where α is a distinguished class in $H^n(K(G,n); G)$.

Note: Also true for $[X, K(G,n)]$ as long as X is connected.

Ex: $K(\underset{G}{\mathbb{Z}}, \underset{n}{1}) = S^1$. $\qquad \langle X, S^1\rangle \cong H^1(X; \mathbb{Z})$

$$(X \xrightarrow[f]{} S^1) \longmapsto f^*(\underbrace{[S^1]^*}_{\substack{\text{class that}\\ \text{evaluates to 1}\\ \text{on fund class}\\ [S^1]}})$$

Now $H^1(X; \mathbb{Z}) \cong \mathrm{Hom}(H_1(X; \mathbb{Z}), \mathbb{Z})$
$\qquad\qquad\qquad \cong \mathrm{Hom}(\pi_1 X, \mathbb{Z})$

Under this ident, our isom $\langle X, S^1\rangle \cong H^1(X; \mathbb{Z})$ is $(f: X \to S^1) \longmapsto (f_*: \pi_1 X \to (\pi_1 S^1 = \mathbb{Z}))$

<u>Reason:</u> $\alpha \in \pi_1 X$ then

$$\left(f^*([S^1]^*)\right)([\alpha]) = [S^1]^*(f_*[\alpha]).$$

In this case, Thm is equivalent to:

- Any $\phi : \pi_1 X \to \mathbb{Z}$ can be realized by some $X \to S^1$
- Any two such realizations are homotopic.

<u>Proof Sketch:</u> X conn CW cplx, $\phi : \pi_1 X \to \mathbb{Z}$ given

Can assume X^0 = one pt, [that way each oriented edge in $X^{(1)}$ is an elt of $\pi_1 X$] Define $f : X^0 \to$ base pt of S^1 and on an edge e^1_α to be something that wraps $\phi([e^1_\alpha])$ times around S^1. The fn f extends over $X^{(2)}$ since ϕ is a homomorphism. Specifically

$$f(\partial d) = \prod f_*[e_{\alpha_i}] = \prod \phi([e_{\alpha_i}]) = \phi\left(\prod [e_{\alpha_i}]\right) = 0$$

and so f extends over ∂d. Extends over higher skelata since $\pi_k S^1 = 0$ for $k > 1$.

Uniqueness up to homotopy is clear on $X^{(1)}$, rest is just homotopy extention prop. ▨

<u>Note</u>: If $f: X \to Y$ preserves base pts, then

$$\begin{tikzcd}
g \circ f & g : Y \to K(G,n) \arrow[l, mapsto, "f^*"'] \\
\langle X, K(G,n) \rangle \arrow[d, "T"'] \arrow[dr, phantom, "\circlearrowleft"] & \langle Y, K(G,n) \rangle \arrow[l, "f^*"'] \arrow[d, "T"] \\
H^n(X;G) & H^n(X;G) \arrow[l, "f^*"]
\end{tikzcd}$$

[where commutativity comes from the last sentence of the thm.]

<u>Proof outline</u>:

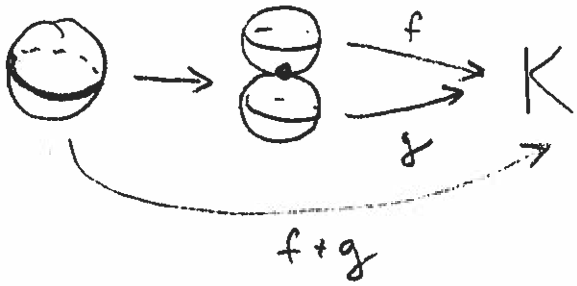
1) Set $h^n(X) = \langle X, K(G,n) \rangle$. This turns out to have a group structure, and so gives a contravariant functor $\left\{ \begin{matrix} \text{Based CW} \\ \text{complexes} \end{matrix} \right\} \to \left\{ \begin{matrix} \text{Abelian} \\ \text{groups} \end{matrix} \right\}$

2) $h^*(X)$ define a reduced cohomology theory where $h^*(S^0) = 0$ for $* > 0$ and $h(S^0) = G$.

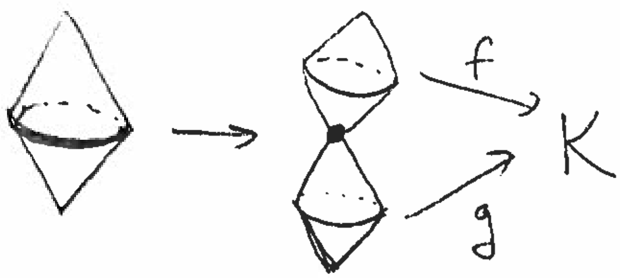
3) Any red. coh theory on CW complexes sat the above is $\cong H^n(-;G)$.

K_n a sequence of spaces. When is $h^n = \langle \cdot , K_n \rangle$ a cohomology theory? [Mention Brown representability.] [Have functoriality, need gp str + long exact.] ④

$\langle X, K \rangle \qquad X = S^n \qquad \langle S^n, K \rangle = \pi_n K$

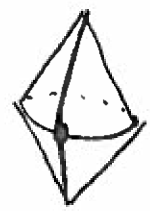


$SX \longrightarrow SX \vee SX$



Reduced suspension:

$$\Sigma X = SX \Big/ \{x_0\} \times I$$



If x_0 = zero cell, then $SX \to \Sigma X$ is a homotopy equivalence.

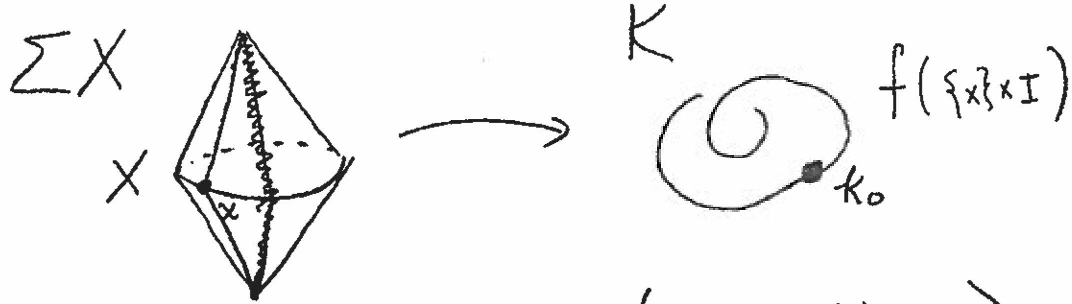
Summary: For any X, $\langle \Sigma X, K \rangle$ has a group str.

Adjoint Relation: Set ΩK = loopspace of K

$$= \left\{ \begin{matrix} \text{maps} \\ I \to K \end{matrix} \;\middle|\; \partial I \to K_0 \text{, the base pt of } K \right\} \subseteq \underbrace{K^I = \{ f : I \to K \}}_{\text{with the compact open topology.}}$$

Base pt of ΩK = (const path at K_0)

Reason:



Thus given $(f : \Sigma X \to K) \longmapsto (x \longmapsto f|_{x \times I}) \in \langle X, \Omega K \rangle$

This is an isomorphism because $\langle \Sigma X, K \rangle$

$= f : X \times I \to K \qquad = \langle X, \Omega K \rangle$

where $(X \times \{0,1\}) \cup (\{x_0\} \times I)$

goes to k_0

$F : X \to \Omega K$

$\left(F(x) = \text{loop in } K\right)(t) \in K$

$t = 0$, $t = 1 \Rightarrow$ base pt

$x_0 \mapsto$ const at k_0 $\Rightarrow \{X_0\} \times I$ goes to k_0.

Compact open topology: Basic open set in $\{f : I \xrightarrow[\text{cont}]{} K\}$

is given by $J \subseteq I$ compact, $U \subseteq K$ open

$$O(J, U) = \{ f : I \to K \mid f(J) \subseteq U \}$$

Useful props: $\pi_{n+1} K = \langle S^{n+1}, K \rangle = \langle S^n, \Omega K \rangle$

$= \pi_n \Omega K$.

So: $\Omega(K(G, n))$ is a $K(G, n-1) \Rightarrow \Omega \mathbb{C}P^\infty$ is homotopy equiv to S^1.

$\pi_0(\Omega^n K) = \pi_n K$