# Lecture 24: Actual computations!

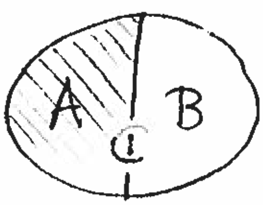
**Def:** $X$ is $n$-connected if $\pi_i(X, x_0) = 0$ for all $i \leq n$.

[0-connected = path connected; 1-conn = simply connected;
Equivalently, every map from $S^i \to X$ is hom. to a
const. map.]

**Def:** $(X, A)$ is $n$-connected if $\pi_i(X, A, x_0) = 0$
for all $x_0 \in A$ and $0 < i \leq n$ and also each
path component of $A$ contains a pt of $A$.

[Rant about excision.]

**Thm:** Let $X$ be a CW complex decomposed as a union of two subcomplexes $A$ and $B$ with $C = A \cap B \neq \emptyset$.
If $(A, C)$ is $m$-connected and $(B, C)$ is $n$-connected, then

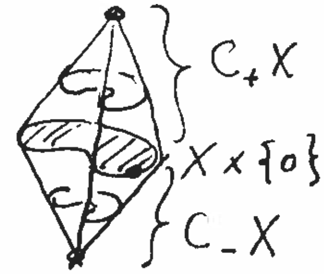


$$\pi_i(A, C) \xrightarrow[i_*]{} \pi_i(X, B) \text{ is an}$$

isomorphism for $i < n+m$ and a
surjection for $i = n+m$.

[Query: What does excision say for homology here?]

$$SX = X \times [-1,1] \Big/ \begin{matrix} X \times \{1\} \\ X \times \{0\} \end{matrix}$$




$$S \cdot S^n = S^{n+1}$$

<u>S functor:</u> $f: X \to Y$ gives $Sf: SX \to SY$

$f \times id_{[-1,1]}$

(from category of top spaces and cont maps to itself)

<u>Leads to:</u>

$$\pi_n(X, x_0) \longrightarrow \pi_{n+1}(SX, x_0 \times \{0\})$$

$\psi$

$(S^n, s_0) \xrightarrow{f} (X, x_0)$ $\qquad$ $(S^{n+1}, s_0 \times \{0\}) \xrightarrow{Sf} (SX, x_0 \times \{0\})$

<u>Freudenthal suspension thm:</u> $X$ is $n-1$ connected.

Then $\pi_i X \xrightarrow[S]{} \pi_{i+1} SX$ is an isom for $i < 2n-1$ and onto for $i = 2n-1$

<u>Cor:</u> $\pi_n S^n \cong \mathbb{Z}$ and is gen by $id_{S^n}$.

The homotopy class of $f: S^n \to S^n$ is determined by its degree.

[Recall def of degree here.]

Pf of Cor: Consider the suspension induced maps:

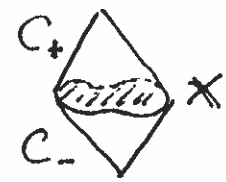
$$\begin{tikzcd}
\mathbb{Z} \cong \pi_1 S^1 \arrow[r, "S"', "\text{onto}"] \arrow[dr, "\text{degree}"', "\cong"] & \pi_2 S^2 \arrow[r, "\cong"] \arrow[d, "\text{degree}"] & \pi_3 S^3 \arrow[r, "\cong"] & \pi_4 S^4 \arrow[r, "\cong"] & \cdots \\
& \mathbb{Z} & & &
\end{tikzcd}$$

gen by id

by F.S.T. since $S^n$ is $(n-1)$ connected.

Since degree: $\pi_2 S^2 \to \mathbb{Z}$ is onto, commutativity implies that $\pi_1 S^1 \longrightarrow \pi_2 S^2$ is injective, hence an isomorphism. ▨

Pf of F.S.T. from excision: Idea: relate $\pi_{i+1}(SX)$ to $\pi_{i+1}(CX, X) \cong \pi_i X$.   Consider:



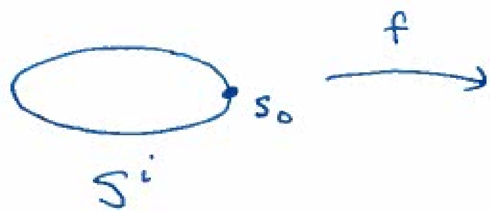

$$\begin{tikzcd}
\pi_i X \arrow[r, bend left, "C_+"] \arrow[rrr, bend right, ""] & \pi_{i+1}(C_+X, X) \arrow[l, "\cong", "\partial"'] \arrow[r, "i_*"'] & (SX, C_-) & \pi_{i+1} SX \arrow[l, "\cong"']
\end{tikzcd}$$

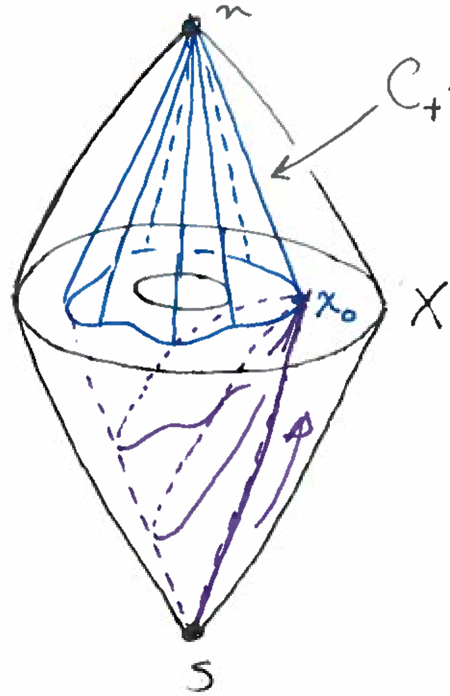
Claim: this is $S_*$

$$\pi_i X \xrightarrow{C_+} \pi_{i+1}(C_+X, X) \quad \text{via } (f : S^i \to X) \mapsto (\underset{\substack{\parallel \\ D^{i+1}}}{CS^i} \to CX)$$

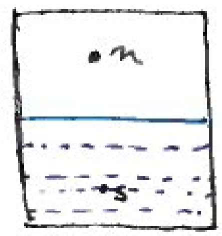
Note that $\partial \circ C_+ = id_{\pi_i X} \Rightarrow C_+$ is the inverse isomorphism to $\partial$.

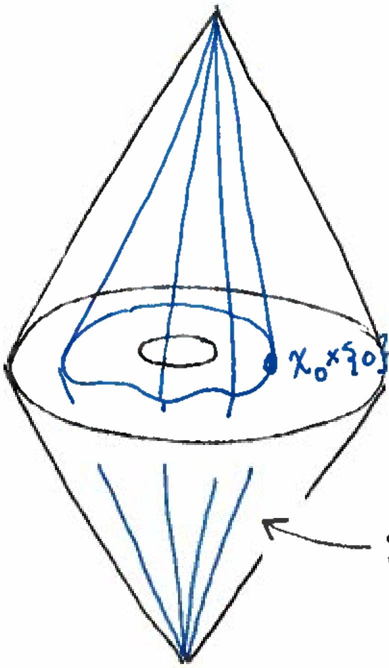
Claim follows from:




$C_+f : (D^{i+1}, S^i, s_0) \to (C_+X, X, x_0)$




$Sf : (D^{i+1}, S^i, s_0) \to (SX, x_0, x_0)$

Now apply excision to $\pi_{i+1}(C_+X, X) \xrightarrow[i_*]{} \pi_{i+1}(SX, C_-X)$

$X$ $n-1$ connected $\Rightarrow$ $(C_+X, X)$ is $n$ connected
$(C_-X, X)$ is $n$ connected

$\Rightarrow$ $i_*$ is an isom for $i+1 < 2n$ and
a surjection for $i+1 = 2n$ $\blacksquare$

**Prop**: $(X, A)$ an $r$-connected CW pair and $A$ is $s$-connected. Then $\pi_i(X,A) \longrightarrow \pi_i(X/A)$ is an isomorphism for $i \leq r+s$ and onto for $i = r+s+1$.


$$X \cup CA \simeq_{h.e.} X \cup CA / CA$$
$$\simeq_{h.e.} X/A$$

Apply excision to $(X,A) \longrightarrow (X \cup CA, CA)$. Extra dimension comes from $(CA, A)$ being $s+1$ connected.