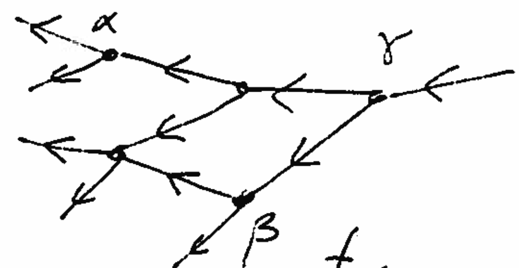
Lecture 17: Direct limits and duality.

Direct limits:

Directed set: I with partial order where $\forall \alpha, \beta \in I$, there is $\gamma \in I$ with $\alpha \le \gamma$ and $\beta \le \gamma$.

Ex: $(\mathbb{N}, \le)$

Ex:



System of groups:

G_α for $\alpha \in I$

$f_{\alpha\beta} : G_\alpha \to G_\beta$

$f_{\alpha\gamma} = f_{\beta\gamma} \circ f_{\alpha\beta}$

$$\begin{tikzcd} G_\alpha \arrow[r, "f_{\alpha\beta}"] \arrow[rr, bend left, "f_{\alpha\gamma}"] & G_\beta \arrow[r, "f_{\beta\gamma}"] & G_\gamma \end{tikzcd}$$

$$\alpha \leftarrow \beta \leftarrow \gamma$$

Ex (A): $I = (\mathbb{N}, \le)$ $\quad G_n = \mathbb{Z}^n$

$$f_{nm} : \mathbb{Z}^n \to \mathbb{Z}^m$$
$$(x_1, \dots, x_n) \longmapsto (x_1, \dots, x_n, 0, \dots, 0)$$

Ex (B): $I = (\mathbb{N}, \le)$

$G_n = \mathbb{Z}/2^n\mathbb{Z}$ $\qquad G_n \to G_{n+1}$

$\qquad\qquad 1 \longmapsto 2.$

$$\varinjlim G_\alpha = \coprod_\alpha G_\alpha \Big/ a \in G_\alpha \sim b \in G_\beta$$

if $\exists \gamma$ with $\alpha, \beta \leq \gamma$ and $f_{\alpha\gamma}(a) = f_{\beta\gamma}(b)$

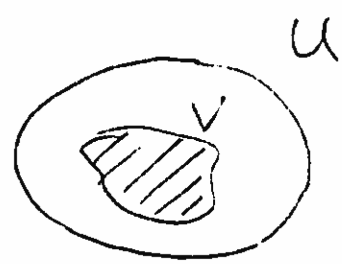
Is a group $[a] + [b] = [f_{\alpha\gamma}(a) + f_{\beta\gamma}(b)]$

<u>Ex A</u>: $\varinjlim \mathbb{Z}^n = \bigoplus_{n=1}^{\infty} \mathbb{Z}$

<u>Ex B</u>: $\varinjlim \mathbb{Z}/2^n\mathbb{Z} = \{ z \in \mathbb{C} \mid z$ a root of unity of order 2^k for some $k \}$

<u>Ex C</u>: $I = \{ U \subseteq \mathbb{R}^n \mid U$ open, $0 \in U \}$

$U \leq V$ if $V \subseteq U$.



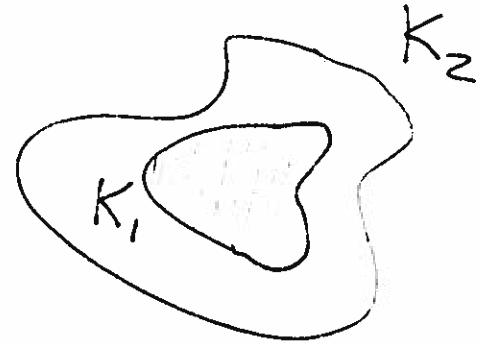
$G_U = C^\infty(U)$ $\quad$ $G_U \to G_V$

$\qquad\qquad\qquad\qquad f \longmapsto f|_V$

$\varinjlim C^\infty(U)$ = germs of smooth funtions at 0.

X space

$I = \{ K \overset{cpt}{\subseteq} X \}$ $\qquad K_1 \le K_2 \Leftrightarrow K_1 \subseteq K_2$



$G_K = H^n(X|K)$

<u>Thm</u>: $\varinjlim_{K \overset{cpt}{\subseteq} X} H^n(X|K) \cong H^n_c(X)$.

$$(X, X \setminus K_2) \longrightarrow (X, X \setminus K_1)$$

$$H^n(X|K_2) \xleftarrow[i^*]{} H^n(X|K_1)$$

<u>Recall</u>:

$$C^n_c(X) = \left\{ \varphi \in C^n(X) \;\middle|\; \begin{array}{l} \exists\, K_\varphi \overset{cpt}{\subseteq} X \text{ with } \varphi(\sigma) = 0 \\ \text{for all } \sigma: \Delta^n \to X \setminus K \end{array} \right\}$$

<u>Pf</u>: Have $H^n(X|K) \xrightarrow[i^*]{} H^n_c(X)$ for each K,

yielding $\varinjlim H^n(X|K) \longrightarrow H^n_c(X)$.

<u>Onto</u>: $[\varphi] \in H^n_c(X)$ is in the image of $H^n(X|K_\varphi)$

<u>1-1</u>: If $[\varphi] = 0$, there is $\psi \in C^{n+1}_c$ with $\varphi = \delta\psi$.

Then $[\varphi] = 0$ in $H^n(X|K_\varphi \cup K_\psi)$

[Don't have to use all cpt sets, just some directed system where every cpt set is contained in some set of the directed system.]

Ex. $X = \mathbb{R}^n \qquad B_k = B_k^{\text{open}}(0)$

$$H_c^*(\mathbb{R}^n) = \varinjlim \underbrace{H^*(\mathbb{R}^n \mid B_k)}_{\tilde{H}^*(\mathbb{R}^n / \mathbb{R}^n \setminus B_k)} = \tilde{H}^*(S^n)$$

Note: H_c^* is not a homotopy invariant, since

$$H_c^i(pt) = \begin{cases} 0 & i > 0 \\ \mathbb{Z} & i = 0 \end{cases}$$

M^n an R-orient. mfld. [Poss. not cpt!] Define

$$D_M : H_c^k(M) \longrightarrow H_{n-k}(M) \quad (\text{all coeff} = R)$$

as follows. If $K^{\text{cpt}} \subseteq M$, by old lemma

$\exists!\ \mu_K \in H_n(M|K)$ s.t. $\mu_K = $ pref. orient in $H_n(M|p)$ for $p \in K$

So have $D_K : H^k(M|K) \longrightarrow H_{n-k}(M)$ ☆

$$\varphi \longmapsto \mu_K \frown \varphi$$

See next page

If $K \subseteq L^{cpt} \subseteq M$ then naturality of $\cap$ means that

$$\begin{tikzcd} H^k(M|K) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & H_{n-k}(M) \arrow[d, "\cong"] \\ H^k(M|L) \arrow[r] & H_{n-k}(M) \end{tikzcd}$$

Taking limits yeilds $H^k_c(M) \to H_{n-k}(M)$

<u>Poincaré Duality</u>: M^n an R-oriented mfld. Then $D_M : H^k_c(M;R) \to H_{n-k}(M;R)$ is an isomorphism.

(★) Why does this make sense?

Let $c \in C_n(M, M \setminus K)$ rep μ_K.

Then $\partial c \subseteq M \setminus K$ where φ is 0.

Hence since $\partial(c \cap \varphi) = (-1)^k(\partial c \cap \varphi - c \cap \delta\varphi)$ $= 0$ in $C_{n-k}(M)$ and so $\mu_K \cap \varphi$ is actually in $H_{n-k}(M)$.