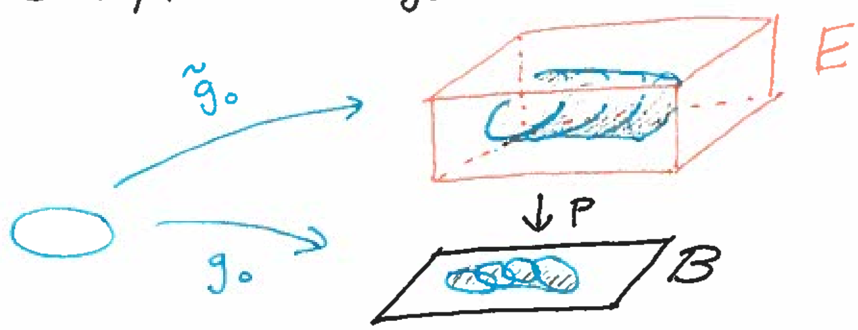
Lecture 29: Fiber bundles.

[Short sequence of spaces leading to a long exact seq in $\pi_* \ldots$]

Homotopy Lifting Property: $p: E \to B$ has H.L.P with respect to X if given $g_0: X \to B$ and a lift every homotopy $g_t: X \times I \to B$ lifts to E starting at $\tilde{g}_0$.

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "\tilde{g}_0"] \arrow[r, "g_0"'] & B \end{tikzcd} \qquad p \circ \tilde{g}_0 = g_0$$



Def: $p: E \to P$ is a fibration if it has F.L.P. with respect to all X.

Ex: $E = B \times F$, p projection onto B. If $\tilde{g}_0 : X \to E$ is given by $\tilde{g}_0(t) = (g_0(t), h(t))$ then define

$$\tilde{g}_t(x) = (g_t(x), h(t)).$$

[Query:] Ex: $p: E \to B$ a covering space.

[In general, $E \to B$ is a "twisted product" with fixed fiber F]

<u>Thm</u>: Suppose $p : E \to B$ is a fibration. Let $b_0 \in B_0$ and $x_0 \in F = p^{-1}(b_0)$. If B is path connected the following is exact

$$\cdots \to \pi_n(F, x_0) \xrightarrow{i_*} \pi_n(E, x_0) \xrightarrow{p_*} \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \cdots \pi_0(F, x_0) \to \pi_0(E, x_0) \to 0.$$

<u>Key claim</u>: $p_* : \pi_n(E, F, x_0) \longrightarrow \pi_n(B, b_0, b_0) = \pi_n(B, b_0)$ is an isomorphism.

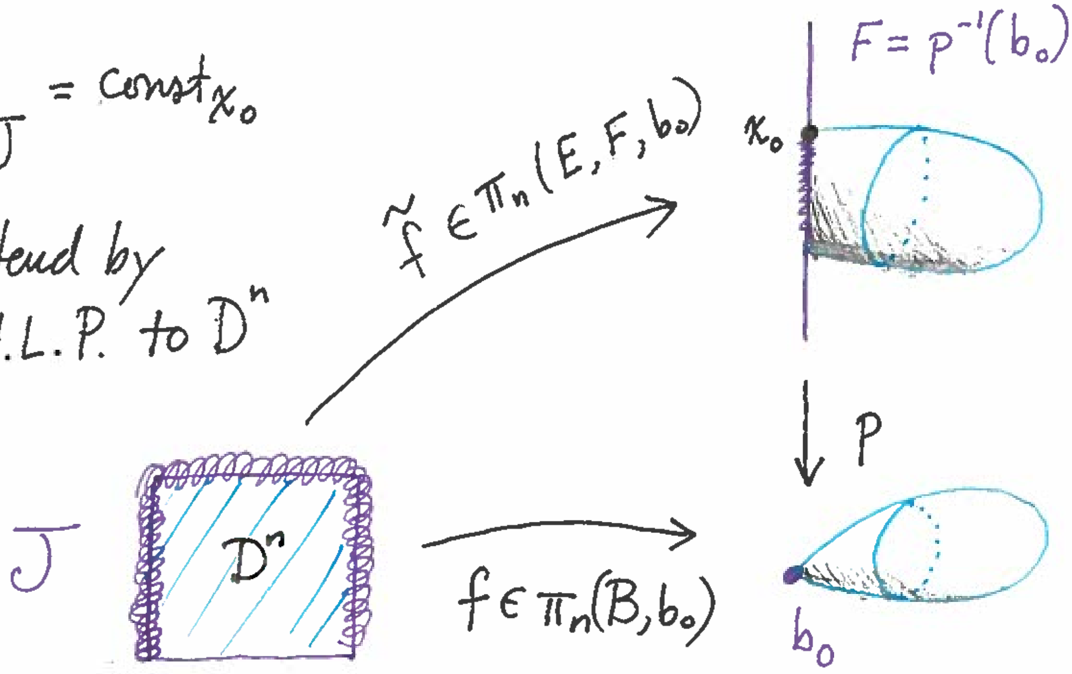
Combining with the long exact sequence of (E, F), this gives the thm except for the very last 0; i.e. every path comp of E contains a point of F.

Given $e \in E$, join $p(e)$ to b_0 by a path, a lift of this path to one starting at e joins e to some pt in F.

<u>Proof Sketch</u>: p_* onto.

$\tilde{f}|_J = \text{const}_{x_0}$

Extend by H.L.P. to D^n



$p_*[\tilde{f}] = [f]$

<u>p_* is 1-1</u>: If $p_*[\tilde{f}] = p_*[\tilde{g}]$ then use HLP to lift homotopy between $p \circ \tilde{f}$ and $p \circ \tilde{g}$ to see $[\tilde{f}] = [\tilde{g}]$. ▨

<u>Fiber Bundles</u>: [Locally a product]

$E \xrightarrow{p} B$ is a fiber bundle with fiber F

total space ↗ ↖ base space

if each pt in B has a nbhd U and a homeomorphism $p^{-1}(U) \xrightarrow{h} U \times F$ where the diagram

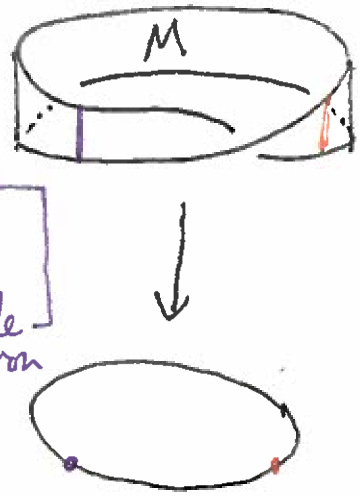
$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "h"] \arrow[dr, "p"'] & & U \times F \arrow[dl, "\text{proj onto } U"] \\ & U & \end{tikzcd}$$

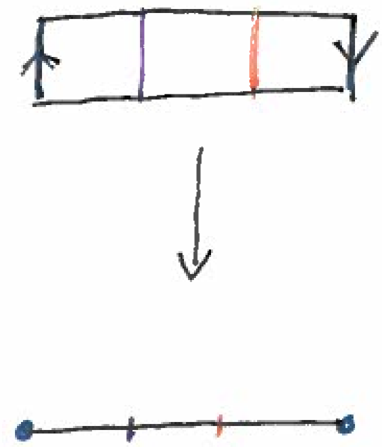
commutes.

<u>Fact</u>: Fiber bundle maps are fibrations.

Ex: 1) $E = B \times F$ 2) Covering space (F = discrete set)

Ex: Möbius band





$\boxed{I \to M \to S^1}$ Notation for fiberbundle or fibration

Ex: Hopf bundle: $S^1 \to S^3 \to S^2$

$$\begin{array}{ccc} S^3 = \{(z_0, z_1) \in \mathbb{C}^2 \mid \underbrace{|z_0|^2 + |z_1|^2 = 1}_{(*)}\} & & (z_0, z_1) \\ \downarrow p & & \downarrow \\ \mathbb{CP}^1 = \mathbb{C}^2 \setminus \{0\} / \mathbb{C}^* & & [z_0 : z_1] \\ \| & & \\ S^2 & & \end{array}$$

For $\lambda \in S^1 \subseteq \mathbb{C}$, note $p(\lambda z_0, \lambda z_1) = p(z_0, z_1)$.

In fact $p^{-1}(pt) = $ circle, since if $(z_0, z_1) \in S^3$ and $(\lambda z_0, \lambda z_1) \in S^3$ then $|\lambda| = 1$ by $(*)$

<u>Local triviality</u>: Take $U \subseteq \mathbb{C}P^1$ to be $\{[z_0 : 1] \mid z_0 \in \mathbb{C}\} \cong \mathbb{C}$

$$h : p^{-1}(U) \longrightarrow U \times S^1 \quad \text{defined by}$$

$$p^{-1}(U) = \{(z_0, z_1) \in S^3 \mid z_1 \neq 0\} \qquad h(z_0, z_1) = \Big(\underbrace{[z_0/z_1 : 1]}_{\text{so diagram commutes.}}, z_1/|z_1|\Big)$$

This is a homeo since here's the inverse:

$$h^{-1}([z_0 : 1], \lambda) = \left(\frac{\lambda z_0}{\sqrt{1+|z_0|^2}}, \frac{\lambda}{\sqrt{1+|z_0|^2}} \right)$$

http://nilesjohnson.net/hopf.html