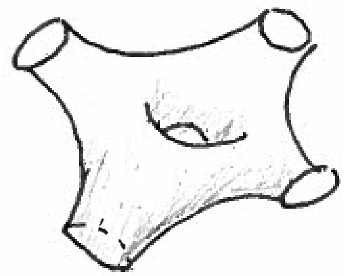
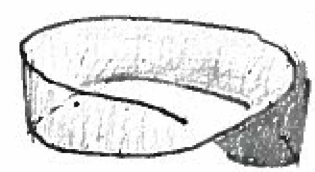
Lecture 19: Other forms of duality.

Poincaré Duality: M^n an R-oriented mfld. Then

$$D_M : H_c^k(M; R) \longrightarrow H_{n-k}(M; R) \text{ is an } \cong \text{ for all } k.$$

A Hausdorff 2nd countable topological space M is a manifold with boundary if every $x \in M$ has an open nbhd $\cong$ to $\mathbb{R}^n$ or $\underbrace{\mathbb{R}^{n-1} \times [0,\infty)}_{\text{pt where this is the case form } \partial M}$.

$n=2$: , , , 

If $\partial M \neq \emptyset$ then $H_n(M) = 0$ since $\underbrace{M \simeq_{h.e.} M \setminus \partial M}_{\text{see below.}}$ and non-cpt mflds have $H_n = 0$.

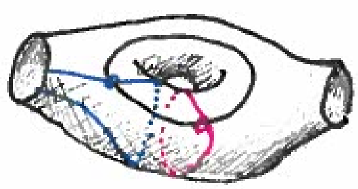
Thm: M^n a cpt R-oriented mfld w/ bdry. Then

Ⓐ $H^k(M) \cong H_{n-k}(M, \partial M)$

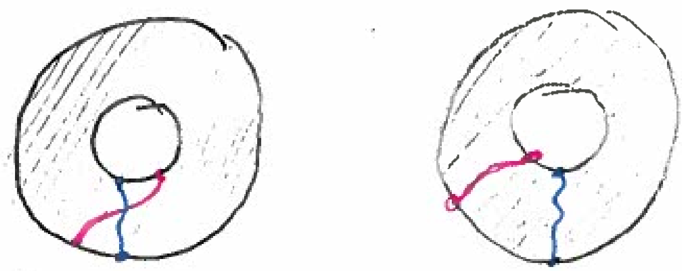
Ⓑ $H^k(M, \partial M) \cong H_{n-k}(M)$

Geometric picture: [cap product on homology]

$$H_1(S) \times H_1(S,\partial S) \to \mathbb{Z} \quad \text{but}$$



not $H_1(S,\partial S) \times H_1(S,\partial S)$ since

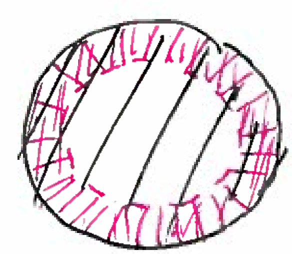


and so $\cap$ product is not well defined. So

$$H^1(S) = \mathrm{Hom}(H_1(S)) \cong H_1(S,\partial S).$$

[Can prove Thm in either way, will use 2nd proof]

Prop: If M is a cpt mfld with bdry, then ∂M has an open nbhd $\cong$ to $\partial M \times [0,1)$

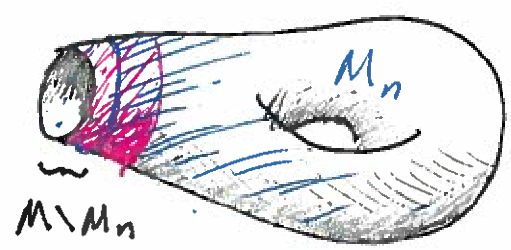


[Pf: See Hatcher.]

Cor: $N = M \setminus (\partial M \times [0,1/2))$ is a def. retract of M and $M \setminus \partial M$. In particular, $\underset{\overset{\cong}{N}}{M} \simeq_{h.e.} M \setminus \partial M$.

Pf of thm: (B) Follows from $H^k(M,\partial M) \cong H^k_c(M \setminus \partial M)$ via the Cor. In more detail, set

$$M_n = M \setminus \left(\partial M \times [0, 1/n] \right)$$



$$H^k_c(M) = \varinjlim_n \underbrace{H^k(M \,|\, M_n)}_{H^k(M,\, M \setminus M_n) \;\cong\; H^k(M_n,\, \partial M_n) \;\cong\; H^k(M,\, \partial M)}$$

Also, clearly $H_{n-k}(M) \cong H_{n-k}(M \setminus \partial M)$.

(A) follows from (B) via long exact sequences of the pair, which are comp. with $[M] \cap \cdot$ ∎

<u>Alexander Duality</u>: If $K \subseteq S^n$ is cpt, locally contractible and not $\emptyset$ or S^n, then for all i:

$$\tilde{H}_i(S^n \setminus K; \mathbb{Z}) \cong \tilde{H}^{n-i-1}(K; \mathbb{Z})$$

<u>Cor</u>: $\tilde{H}_*(S^n \setminus K)$ does <u>not</u> depend on how K is embedded in S^n.

$S^1 \hookrightarrow S^3$:  or

[Put up beforehand]

$S^1 \to$

$\pi_1(S^3 \setminus K)$ — is not finitely generated.

In each case, $\tilde{H}^*(S^3 \setminus S^1) = \begin{cases} \mathbb{Z} & i=1 \\ 0 & \text{otherwise} \end{cases}$

Proof sketch: Will show [Skip if low on time]

$$H_i(S^n \setminus K) \cong H_c^{n-1}(S^n \setminus K) \qquad \text{[Poincaré]}$$

$$\cong \varinjlim_{\substack{U \supseteq K \\ \text{open}}} \underbrace{H^{n-i}(S^n - K, U - K)}_{H^{n-i}(S^n \setminus K \mid S^n \setminus U)} \qquad \text{[local co as } \varinjlim \text{]}$$

Query: what is the directed system here?

$$\cong \varinjlim H^{n-i}(S^n, U) \qquad \text{[Excision.]}$$

$$4 \quad \cong \varinjlim \tilde{H}^{n-i-1}(U) \quad \text{if } i \neq 0$$

$$5 \quad \cong \tilde{H}^{n-i-1}(K)$$

Step 4 is the long exact sequence of the pair (S^n, U):

$$\leftarrow \underset{0}{\tilde{H}^{n-i}(S^n)} \leftarrow H^{n-i}(S^n, U) \xleftarrow{\cong} \tilde{H}^{n-i-1}(U) \leftarrow \underset{0}{\tilde{H}^{n-i-1}(S^n)}$$

using $i=0$ here.

For part 5, need a little point-set topology. 5

If K has an open nbhd U which def retracts to K, then this is easy.

In general only have retractions but argument can be made to work.