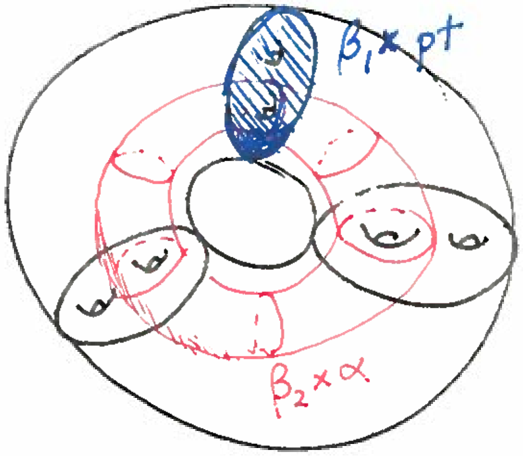
Think about homological case: $n=1$

Y



$Y \times S^1$

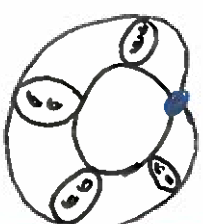
$\beta_1 \times pt + \beta_2 \times \alpha$.

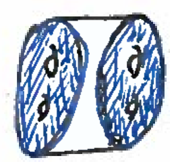
<u>Pf of Lemma:</u> Note that $H^{n+1}(Y) \xrightarrow{p^*} H^{n+1}(Y \times S^1) \xrightarrow{i} H^{n+1}(Y)$

p^* is 1-1 since the composition: $\beta_1 \longmapsto \beta_1 \times 1$

$$Y \xrightarrow{i} (Y \times \{pt\}) \subseteq Y \times S^1 \xrightarrow{p} Y$$

is the identity. Using the long exact seq for the pair $(Y \times S^1, Y \times \{pt\})$ shows that it is enough to understand

$$Y \times S^1 / Y \times pt =$$  $$\cong Y \times I / Y \times \partial I$$



and prove that $H^n(Y) \longrightarrow H^{n+1}(Y \times I, Y \times \partial I)$

$$\beta \longmapsto \beta \times \alpha$$

is an isomorphism, where $\alpha \in H^1(I, \partial I) \cong H^1(S^1)$

Claim ☆

First:
$\partial I = \{0,1\}$ so $H^0(\partial I) = R^2 = \langle 1_0 , 1_1 \rangle$

$$\begin{tikzcd} & & R & R & 0 \arrow[d, phantom, "\parallel"] \\ 0 & H^1(I,\partial I) \arrow[l] & H^0(\partial I) \arrow[l, "\delta"'] & H^0(I) \arrow[l] & H^0(I,\partial I) \arrow[l] \arrow[d, phantom, "\cong"] \\ & & 1_0 + 1_1 & 1_I \arrow[l, mapsto] & \tilde{H}^0(I/\partial I) \end{tikzcd}$$

$\delta(1_0) = (\text{unit in } R)\,\alpha$

Second: [Exercise] More generally, assume $A \subseteq X$ is reasonable, and consider

$$\begin{tikzcd} H^k(A) \oplus H^l(Y) \arrow[r, "\delta \oplus id"] \arrow[d, "\times"'] & H^k(X) \oplus H^{l+1}(Y) \arrow[d, "\times"] \\ H^{k+l}(A \times Y) \arrow[r, "\delta"] & H^{k+l+1}(A \times Y, X \times Y) \end{tikzcd}$$

$$(\varphi, \psi) \longmapsto (\delta\bar{\varphi}, \psi) \longmapsto (p_X^\#(\delta\bar{\varphi})) \cup (p_Y^\# \psi)$$

$$(\varphi, \psi) \longmapsto p_A^\#(\varphi) \cup p_Y^\#(\psi) \longmapsto \delta\left(p_X^\#(\bar{\varphi}) \cup p_Y^\#(\psi)\right)$$

$$(p_X^\#(\delta\bar{\varphi})) \cup (p_Y^\# \psi) = \delta\left(p_X^\#(\bar{\varphi}) \cup p_Y^\#(\psi)\right) \quad (\dagger)$$

Reason: Start with cocycles $\varphi \in C^k(Y)$ and $\psi \in C^l(A)$. Extend φ to $\bar{\varphi}$ in $C^k(X)$. Conclude the equality $(\dagger)$ since

$$\delta\left(p_X^\#(\bar{\varphi}) \cup p_Y^\#(\psi)\right) = \left(\delta p_X^\#(\bar{\varphi})\right) \cup p_Y^\#(\psi)) + \underbrace{(-1)^k \bar{\varphi} \cup \delta\psi}_{=0.}$$

$$\beta\times(1_0+1_1) = (\beta,\beta) \longleftarrow\!\shortmid\ \beta$$

$$\begin{tikzcd}
& 0 & H^n(Y\times\{0\})\oplus H^n(Y\times\{1\}) & H^n(Y) & 0 \arrow[dl] & \\
\arrow[l] & H^{n+1}(Y\times I, Y\times\partial I) \arrow[l] \arrow[u] & H^n(Y\times\partial I) \arrow[l, "\delta"'] \arrow[u, phantom, "\cong" sloped] & H^n(Y\times I) \arrow[l] \arrow[u, phantom, "\cong" sloped] & H^n(Y\times I, Y\times\partial I) \arrow[l] \arrow[u] & \arrow[l] \\
& H^n(Y)\oplus H^1(I,\partial I) \arrow[u, "\times"] & H^n(Y)\oplus H^0(\partial I) \arrow[l, "\mathrm{id}\oplus\delta"] \arrow[u, "\times"', "\cong"] & & &
\end{tikzcd}$$

exercise

This map is just the diagonal map hence injective.

Note that $\delta\big|_{H^n(Y\times\{0\})} = \delta\big|_{H^n(Y)\times 1_0}$ is an isomorphism from $H^n(Y) \longrightarrow H^{n+1}(Y\times I, Y\times\partial I)$. By commutativity, we have $\delta(\beta\times 1_0) = \beta\times\delta(1_0) = (\text{fixed unit in } R)\,\beta\times\alpha$.

This proves claim ✪ and thus the lemma.

Lecture 7: Last time and first half of class: see "Lecture 6" notes pgs 4-6.

HW#2: Due Wed Sept 24.

Hatcher:

and others to be assigned.

What is $H^*(X \times Y)$? Starting pt:

$$H^*(X) \times H^*(Y) \xrightarrow[\times]{} H^*(X \times Y)$$

$$\alpha \quad\quad \beta \longmapsto \alpha \times \beta = p_X^*(\alpha) \cup p_Y^*(\beta)$$

[Might hope this is an isomorphism, but...]

$X = S^1 \quad Y = \{pt\} \quad\quad X \times Y = S^1$

$$(\mathbb{Z}_{(0)} \oplus \mathbb{Z}_{(1)}) \oplus \mathbb{Z}_{(0)} \longrightarrow \mathbb{Z}_{(0)} \oplus \mathbb{Z}_{(1)}$$

Also, $\times$ is bilinear, not a homomorphism. That is

$(\alpha_1 + \alpha_2) \times \beta = \alpha_1 \times \beta + \alpha_2 \times \beta$ and reversed and so

$$\begin{aligned} \times((\alpha_1, \beta_1) + (\alpha_2, \beta_2)) &= \times((\alpha_1 + \alpha_2, \beta_1 + \beta_2)) \\ &= \alpha_1 \times \beta_1 + \alpha_1 \times \beta_2 + \alpha_2 \times \beta_1 + \alpha_2 \times \beta_2 \\ &\neq \times((\alpha_1, \beta_1)) + \times((\alpha_2, \beta_2)) \end{aligned}$$