Lecture 7: Last time and first half of class: see "Lecture 6" notes pgs 4–6.

HW#2: Due Wed Sept 24.

Hatcher:

and others to be assigned.

What is $H^*(X \times Y)$? Starting pt:

$$H^*(X) \times H^*(Y) \xrightarrow{\times} H^*(X \times Y)$$
$$\alpha \quad \beta \longmapsto \alpha \times \beta = p_X^*(\alpha) \cup p_Y^*(\beta)$$

[Might hope this is an isomorphism, but...]

$X = S^1 \quad Y = \{pt\} \qquad X \times Y = S^1$

$$(\mathbb{Z}_{(0)} \oplus \mathbb{Z}_{(1)}) \oplus \mathbb{Z}_{(0)} \longrightarrow \mathbb{Z}_{(0)} \oplus \mathbb{Z}_{(1)}$$

Also, $\times$ is bilinear, not a homomorphism. That is

$(\alpha_1 + \alpha_2) \times \beta = \alpha_1 \times \beta + \alpha_2 \times \beta$ and reversed and so

$$\begin{aligned} \times\big((\alpha_1, \beta_1) + (\alpha_2, \beta_2)\big) &= \times\big((\alpha_1 + \alpha_2, \beta_1 + \beta_2)\big) \\ &= \alpha_1 \times \beta_1 + \alpha_1 \times \beta_2 + \alpha_2 \times \beta_1 + \alpha_2 \times \beta_2 \\ &\neq \times((\alpha_1, \beta_1)) + \times((\alpha_2, \beta_2)) \end{aligned}$$

Solution: Replace $\times$ with $\otimes$.

A, B abelian gps

$$A \otimes B = \left\{ \begin{matrix} \text{gp gen by} \\ a \otimes b \end{matrix} \right\} \cong \bigoplus_{(a,b) \in A \times B} \mathbb{Z}[a \otimes b] \Big/ \begin{matrix} (a+a') \otimes b = a \otimes b + a' \otimes b \\ a \otimes (b+b') = a \otimes b + a \otimes b' \end{matrix}$$

That is, quotient $\bigoplus_{(a,b) \in A \times B} \mathbb{Z}[a \otimes b]$ by the subgp

gen by $(a+a') \otimes b - a \otimes b - a' \otimes b$

all $a \otimes (b+b') - a \otimes b - a \otimes b'$.

Ex: $A = \underset{a_1}{\mathbb{Z}} \oplus \underset{a_2}{\mathbb{Z}}$ $\quad A \otimes B \cong \mathbb{Z}^6$ with basis $\{a_i \otimes b_j\}$

$B = \underset{b_1}{\mathbb{Z}} \oplus \underset{b_2}{\mathbb{Z}} \oplus \underset{b_3}{\mathbb{Z}}$ $\quad$ typical elt:

$$3a_1 \otimes b_2 + 5a_1 \otimes b_3 + 2a_2 \otimes b_1$$

Key: $\varphi : A \times B \to C$ is bilinear, get a homomorphism $\bar{\varphi} : A \otimes B \to C$

$$a \otimes b \mapsto \varphi(a,b)$$

Conversely, a homomorphism $\psi : A \otimes B \to C$

gives a bilinear map $\tilde{\psi}: A \times B \longrightarrow C$

via $\tilde{\psi}((a,b)) = \psi(a \otimes b)$.

[$A \otimes B$ is the "smallest" and so "universal" thingy for which this is true.]

Consider the homomorphism where coeffs are in some ring R.

$$H^*(X) \otimes H^*(Y) \xrightarrow{\times} H^*(X \times Y)$$
$$a \otimes b \longmapsto a \times b$$

If define a mult by

$$(a \otimes b) \cdot (c \otimes d) = (-1)^{|b||c|} (a \cup c) \otimes (b \cup d)$$

then this map is a ring homomorphism.

Thm: If X and Y are CW complexes and $H^*(Y)$ is a free R-module then $\times$ is an isom.

Cor: Always applies when $R =$ Field.

For general case, see Section 3.B.

Involves Tor.

<u>Division algebra</u>: $\mathbb{R}^n \times \mathbb{R}^n \xrightarrow[\text{bilinear}]{} \mathbb{R}^n$ and $\forall a \neq 0, b$ in $\mathbb{R}^n$ both $ax = b$ and $xa = b$ are solvable.

Equivalently, no zero divisors.

[Not assuming comm, assoc, unital, ...]

<u>Ex</u>: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H} = \mathbb{R}^4 = \langle 1, i, j, k \rangle$, $\mathbb{O} \cong \mathbb{R}^8$

Quaterions $\quad i \cdot j = k$, $i^2 = j^2 = -1$, $ji = -ji$ $\quad$ [not associative]

<u>Thm</u>: If $\mathbb{R}^n$ has the structure of a division algebra, then $n = 2^k$. [In fact, there is only $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.]

<u>We'll need</u>:

$$H^*(\mathbb{RP}^n; \mathbb{F}_2) = \mathbb{F}_2[\alpha]/(\alpha^{n+1})$$

$\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$

α is the gen. of $H^1(\mathbb{RP}^n) \cong \mathbb{F}_2$

[See text and/or wait a week or two.]

$$H^*(\mathbb{RP}^n \times \mathbb{RP}^n; \mathbb{F}_2) = \mathbb{F}_2[\alpha, \beta]/(\alpha^{n+1}, \beta^{n+1})$$

<u>Künneth</u>:

$$H^n(X \times Y) = \bigoplus_{k=0}^{n} H^k(X) \otimes H^{n-k}(Y)$$

$X = \mathbb{R}P^n \qquad Y = \mathbb{R}P^n \qquad$ α gen $H^1(X)$ $\quad 1_X$ gen of $H^0(X)$

β gen $H^1(Y)$ $\quad 1_Y$ gen of $H^0(Y)$

$$H^1(X \times Y) \cong \underbrace{H^1(X) \otimes H^0(Y)}_{\substack{\cong \mathbb{F}_2 \text{ gen by} \\ \alpha \otimes 1_Y}} \oplus \underbrace{H^0(X) \otimes H^1(Y)}_{\text{gen by } 1_X \otimes \beta}$$

$= \mathbb{F}_2^2$ with basis $\alpha \times 1_Y$ and $1_X \times \beta$

$$H^2(X \times Y) = \langle \alpha^2 \times 1, \alpha \times \beta, 1 \times \beta^2 \rangle = \mathbb{F}_2^3$$

$\vdots$

etc.

<u>Pf of thm:</u> Set $g : S^{n-1} \times S^{n-1} \longrightarrow S^{n-1}$ to be $g(x,y) = x \cdot y / |xy|$ [Makes sense because no 0-divisors]

As $(-x) \cdot y = -(x \cdot y) = x(-y)$ we have

$$g(-x, y) = -g(x,y) = g(x,-y).$$

So get a map $h : P^{n-1} \times P^{n-1} \longrightarrow P^{n-1}$ $\quad [P = \mathbb{R}P^{n-1}]$

<u>Claim</u>: With $\mathbb{F}_2$ coeff's, $H^1(P^{n-1} \times P^{n-1}) \xleftarrow{h^*} H^1(P^{n-1})$

$$\alpha + \beta \longleftarrow\!\shortmid \gamma \leftarrow \text{the gen}$$

$\alpha = \gamma \times 1$, $\quad \beta = 1 \times \gamma$

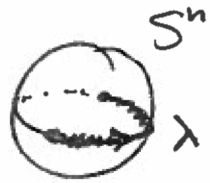
[**Note**: Because of cup product, this completely determines

$$H^*(P^{n-1} \times P^{n-1}) \longleftarrow H^*(P^{n-1})$$
]

Proof of Claim: Take $n > 2$ so $\pi_1 P^{n-1} = \mathbb{Z}/2\mathbb{Z}$ [$= H_1(P^{n-1}; \mathbb{Z})$.

Let's compute $\pi_1(P^{n-1} \times P^{n-1}) \xrightarrow{h_*} \pi_1(P^{n-1})$

$$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2$$

What's image of $(1,0)$? 

gives a gen of $\pi_1 P^{n-1}$

As loop this is $(\lambda, \mathrm{const}_{y_0}) \xmapsto[h_*]{} \lambda \cdot y_0 / |\lambda \cdot y_0|$

Effectively, changing λ by a linear trans $- \cdot y_0$, still get something gen $\pi_1(P^{n-1})$. So $h_*(1,0) = 1$
$h_*(0,1) = 1$.

Same on $H_1(P^{n-1}; \mathbb{Z} \text{ or } \mathbb{F}_2)$ and dualizing gives the claim. ▨

Proof of thm: next time.

<u>Rmk</u>: Just like $\mathbb{R}P^n$ and $\mathbb{C}P^n$, there is also $\mathbb{H}P^n$ and $\mathbb{O}P^1$ and $\mathbb{O}P^2$.

dim/degree.

$$H^*(\mathbb{H}P^n) = \mathbb{Z}[\alpha]/\alpha^{n+1} \qquad |\alpha| = 4.$$

$$H^*(\mathbb{O}P^2) = \mathbb{Z}[\alpha]/\alpha^3 \qquad |\alpha| = 8$$

Reason no $\mathbb{O}P^n$ is need assoc. of mult to define a projective space in general.