Lecture 11: Orientations and $H_n(M;\mathbb{Z})$

$H_n(X|A) = H_n(X, X-A)$ (local homology)

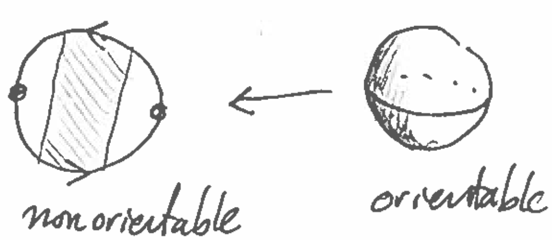
M an n-mfld. A local orient of M at x is a gen $\mu_x \in H_n(M|x;\mathbb{Z}) \cong \mathbb{Z}$. An orientation of M is a fn $x \mapsto \mu_x$ s.t. $\forall\ B^{\text{bounded}}_{\text{ball}} \subseteq \mathbb{R}^n \subseteq U^{\text{open}} \subseteq M$ there is a $\mu \in H_n(X|B)$ so that $\forall x \in B$ one has:

$$H_n(M|B) \xrightarrow{i} H_n(M|x)$$
$$\mu \longmapsto \mu_x$$

[Also put up thm on page 3 as a goal.]

Generator $[\sigma]$ of $H_n(M|x) = H_n(M, M\setminus\{x\})$

Motivation: $\mathbb{R}P^2$ S^2



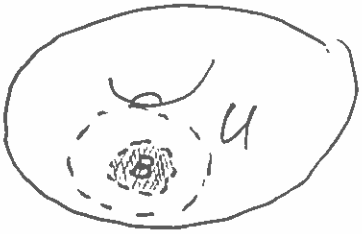
HW due next wed:

Define for any n-mfld

$$\tilde{M} = \{ \mu_x \mid x \in M,\ \mu_x \text{ a local orient at } x \}$$

topologized via $\forall B^{\text{bd}}_{\text{ball}} \subseteq \mathbb{R}^n_{\text{open}} \subseteq M$ and gen $\mu_B \in H_n(M|B)$ we declare the following to be open

$$U(\mu_B) = \{ \mu_x \in \tilde{M} \mid x \in B \text{ and } \underset{H_n(M|B) \longrightarrow H_n(M|x)}{\mu_B \longmapsto \mu_x} \}$$



Consider $p : \tilde{M} \longrightarrow M$, which
$\qquad\qquad\quad \mu_x \longmapsto x$

it is easy to check is a covering map. [Q: Degree = 2]

<u>Prop</u>: $\tilde{M}$ is orientable, via taking as the orientation

$$\tilde{\mu}_{\mu_x} \in H_n(\tilde{M}|\mu_x) \cong H_n(U(\mu_B)|\mu_x) \xrightarrow[\cong]{p_*} H_n(B|x)$$

$$\tilde{\mu}_{\mu_x} \longleftarrow\!\!\shortmid\ \mu_x$$

<u>Prop</u>: Suppose M is connected. Then M is orientable iff $\tilde{M}$ has connected components.

<u>Cor</u>: If $\pi_1 M = 1$, then M is orientable. $[S^n, \mathbb{C}P^n]$

Define with analogous topology the larger covering space

$$\begin{tikzcd} M_{\mathbb{Z}} \arrow[d, "p"] \\ M \arrow[u, bend left, dotted, "s"] \end{tikzcd} \qquad M_{\mathbb{Z}} = \{ \alpha_x \in H_n(M|x) \mid x \in M \} \supseteq \tilde{M}$$

[Can define for any ring.]

For orient M, this is $M \times \mathbb{Z}$. A <u>section</u> of $M_{\mathbb{Z}} \xrightarrow{p} M$ is a cont map $s : M \to M_{\mathbb{Z}}$ where $p \circ s = id_M$.

<u>Ex</u>: $s : x \longmapsto 0 \in H_n(M|x)$.

<u>Thm</u>. M^n closed connected. Then ⓐ $H_k(M;\mathbb{Z}) = 0 \ \forall k > n$.

ⓑ If M is orientable, then $H_n(M;\mathbb{Z}) \longrightarrow H(M|x;\mathbb{Z})$ is an isom $\forall x \in M$. In particular, $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$.

ⓒ Otherwise $H_n(M;\mathbb{Z}) = 0$.

<u>Lemma</u>: $A^{cpt} \subseteq M^n$. Then

① If $x \mapsto \alpha_x$ is a section of $M_{\mathbb{Z}} \to M$, then $\exists! \, \alpha_A \in H_n(M|A)$ whose image in $H_n(M|x)$ is α_x for all $x \in A$.

② $H_k(M|A) = 0$ for all $k > n$.

Lemma $\Rightarrow$ Thm: Part ⓐ follows from ② with $A = M$.

Let $\Gamma(M)$ be the set of sections of $M_{\mathbb{Z}} \to M$, which is a $\mathbb{Z}$-module. Have a homomorphism

$$H_n(M) \longrightarrow \Gamma(M) \quad \text{by} \quad \alpha \longmapsto \left(x \longmapsto i(\alpha) \in H_n(M|x)\right)$$

By ①, this is an isomorphism. Since M is connected, a section is det by its value at some fixed $p \in M$. When M is orient, $M_{\mathbb{Z}} = M \times \mathbb{Z}$ so

$H_n(M) = \mathbb{Z}$ and each $H_n(M) \to H_n(M|x)$ is an isom.

If M is non orient, the only section is the zero section. So $H_n(M) = 0$. ▨

Outline of Pf of Lemma:

① [Key] True for $A, B, A \cap B \Rightarrow$ true for $A \cup B$.

② Suffices to consider $M = \mathbb{R}^n$.

③ Holds for convex $A \subseteq \mathbb{R}^n$, hence unions of such

④ Step 3 $\Rightarrow$ holds for all cpt $\subseteq \mathbb{R}^n$

Note that 1 implies that if true for $\{A_i^{cpt}\}_{i=1}^{m}$ and all $A_{i_1} \cap \cdots \cap A_{i_k}$ true for $\bigcup_{i=1}^{m} A_i$.

For 2, by cptness any $A \overset{cpt}{\subseteq} M$ is a finite union $\bigcup_{i=1}^{m} A_i$ where each $A_i \subseteq U_i$ open $\cong \mathbb{R}^n$.

By excision, $H_n(M | A_i) \cong H_n(U_i | A_i)$ and also for any intersection $A_i \cap \{\text{others}\}$. So if true for cpt $A \subseteq \mathbb{R}^n$ done.

3. If $A \subseteq \mathbb{R}^n$ is cpt convex, then pick $x \in A$. Both $\mathbb{R}^n - A$ and $\mathbb{R}^n - \{x\}$ def. retract to a large sphere centered at x.

Hence $H_*(\mathbb{R}^n | A) \to H_*(\mathbb{R}^n | x)$ is an isom.

[Query: Where did I use convexity?]