Lecture 14: Poincaré Duality via triangulations.

Cap Product: $H_k(X) \times H^\ell(X) \longrightarrow H_{k-\ell}(X)$ for $\ell \leq k$.

$\sigma : \Delta^k \to X \qquad \varphi \in C^\ell(X)$

$$\sigma \cap \varphi = \varphi(\sigma|_{[v_0,\dots,v_\ell]})\, \sigma_{[v_\ell,\dots,v_k]}$$

Poincaré Duality: M is R-orientable, $[M] \in H_n(M)$ a generator. Then $D : H^k(M) \longrightarrow H_{n-k}(M)$

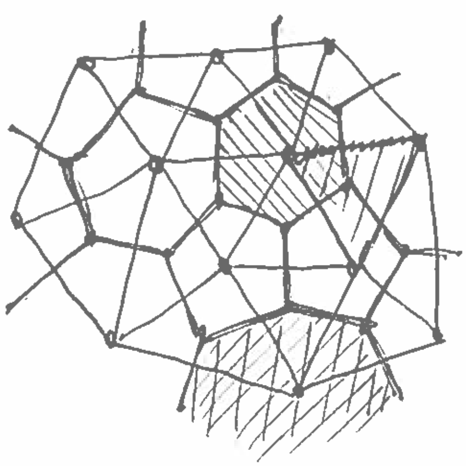
$$\varphi \longmapsto [M] \cap \varphi$$

is an isomorphism.

[Will give two proofs... starting with one that Poincaré would recognize...]

Let M be a clsd n-mfld with a triangulation $\mathcal{T}$.

Dual cell decomposition $\mathcal{D}$



k-simplex in $\mathcal{T}$ $\qquad$ $n-k$ cell in $\mathcal{D}$

$$\sigma \longleftrightarrow \hat{\sigma}$$

Reverses inclusion relations

$$\sigma_0 < \sigma_1 \Longrightarrow D(\sigma_0) > D(\sigma_1)$$

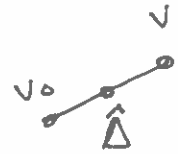
$<$ is a subsimplex/cell

In the end, we'll get an isomorphism

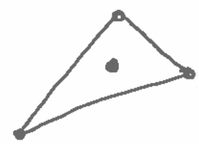
$$\text{cohomology complex of } \mathcal{T} = C^* \longrightarrow D_* = \text{homology complex w.r.t. } \mathcal{D}$$

of chain complexes, proving Poincaré Duality.

Consider a simplex



$$\Delta = [v_0, v_1, \ldots, v_n] \subseteq \mathbb{R}^m$$



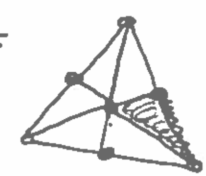
The barycenter of Δ is $\hat{\Delta} = \frac{1}{n+1} \sum v_i$

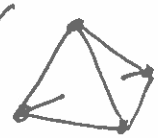
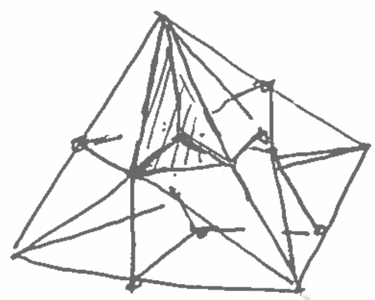
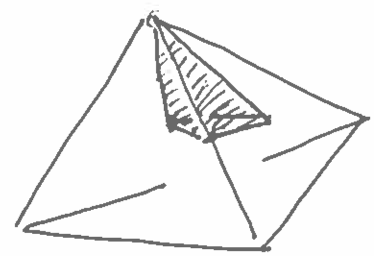
The barycentric subdivision $sd(\Delta)$ of Δ is

$$\Delta = \bigcup \left\{ [\hat{\Delta}, w_0, \ldots, w_{n-1}] \;\middle|\; \begin{array}{c}[w_0, \ldots, w_{n-1}] \\ \text{is a simplex in} \\ sd(\partial\Delta)\end{array} \right\}$$

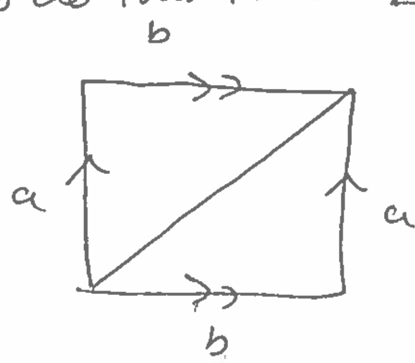
$sd(\cdot) = \cdot$

sd(—) = 

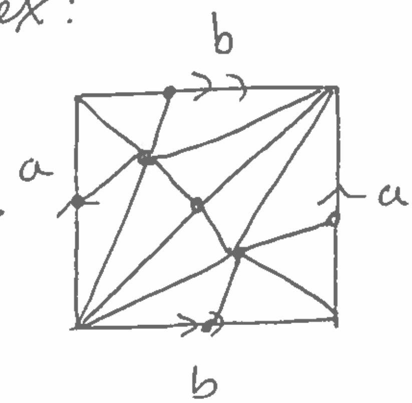
$sd(\Delta) =$ 

sd() =  

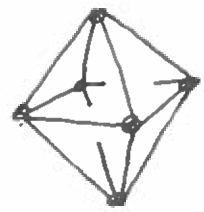
Can also do this to a Δ-complex!



$\longrightarrow$



A Δ-complex X is <u>simplicial</u> if any subset $\{v_0, \dots, v_n\}$ of $X^{(0)}$ is the vertices of <u>at most</u> one n-cell in X:

<u>Yes</u>: 

<u>No</u>: 

[Go back to torus exp.]

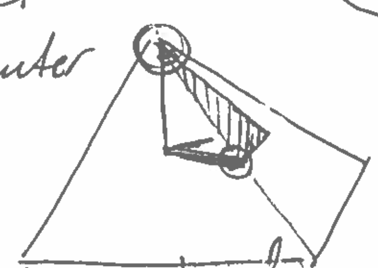
<u>Lemma</u>: Any Δ complex can be made simplical by subdividing <u>twice</u>.

[A simplical Δ-complex is more usually called a simplical complex.]

Let $\mathcal{T}$ be a simplical complex structure on M^n consist of Δ^n's glued along faces. Each simplex σ in $sd(\mathcal{T})$ has vertices which we order $[\hat{\alpha}_{i_1}, \hat{\alpha}_{i_2}, \dots, \hat{\alpha}_{i_k}]$ ☆

$\alpha_{i_1} > \alpha_{i_2} > \dots > \alpha_{i_k}$

Here, the "last vertex" is the barycenter of the lowest dim'l simplex involved.



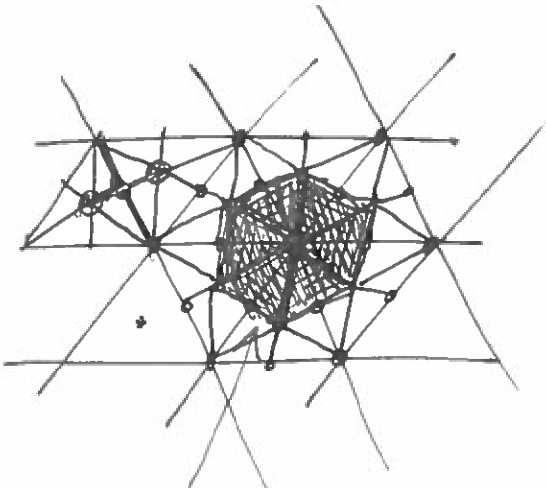
For σ in $\mathcal{T}$ define

$$D(\sigma) = \bigcup \{ \operatorname{int}(\alpha) \mid \alpha \subseteq sd(\mathcal{T}) \text{ with } \hat{\sigma} \text{ as the last vertex.} \}$$

$\bar{D}(\sigma)$ = closure of $D(\sigma)$

$= \bigcup \{\alpha \subseteq sd(\mathcal{J}) \mid \text{last vertex } \hat{\sigma}\}$

$\mathring{D}(\sigma) = \bar{D}(\sigma) - D(\sigma)$



$\bar{D}$(vertex)

<u>Lemma</u>: (a) The $D(\sigma)$ are disjoint and their union is M

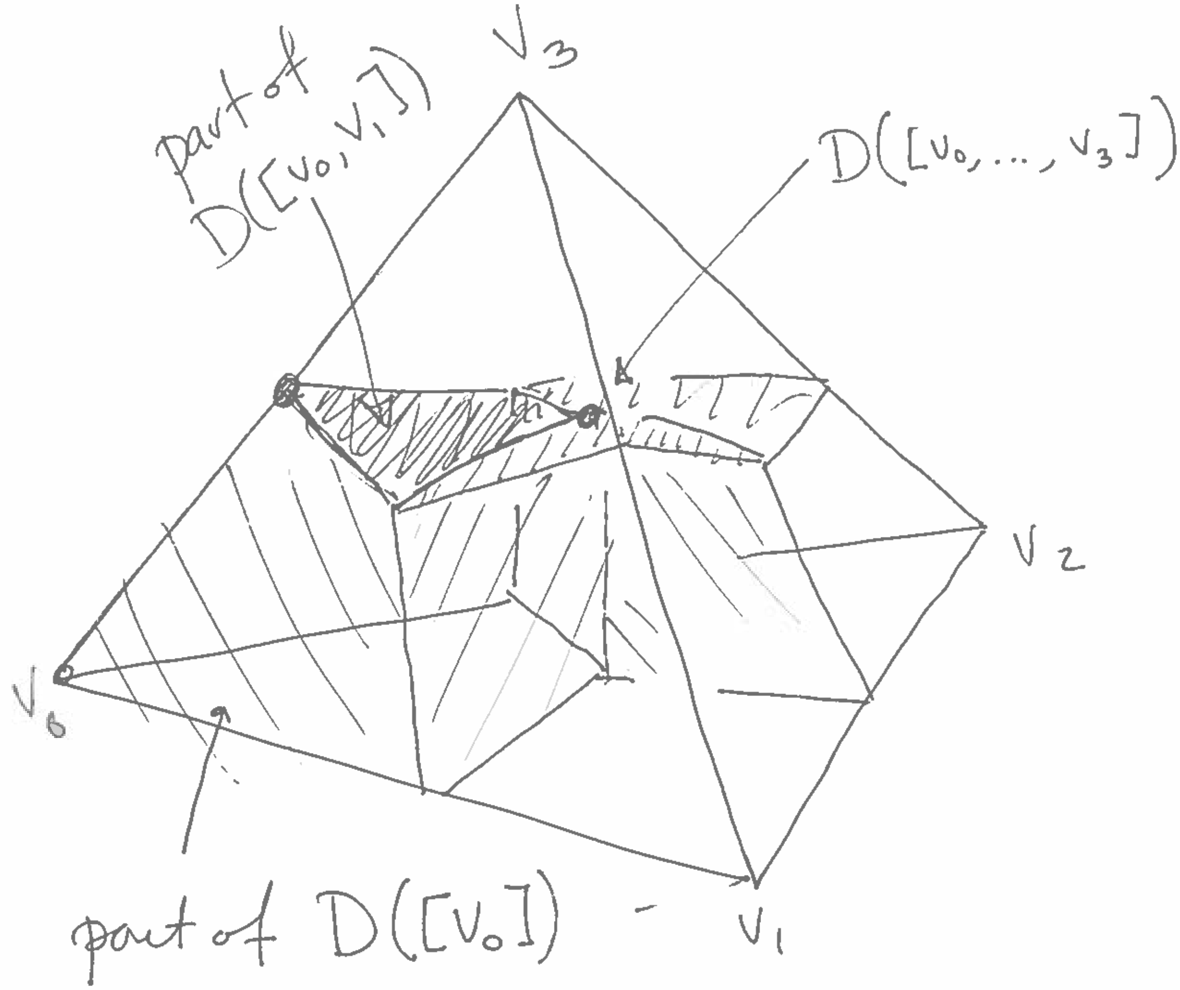
(b) $\bar{D}(\sigma)$ is a subcomplex of $sd(\mathcal{J})$ of dim $n-k$ where $|\sigma| = k$.

(c) $\dot{D}(\sigma) = \{D(\tau) \mid \tau \gneq \sigma\}$

<u>Pf</u>: (a) Every α in $sd(\mathcal{J})$ has a unique last vertex.

(b) If $|\sigma| = K$, then in some Δ^n of $\mathcal{J}$ and $\alpha \subseteq \bar{D}(\sigma)$ can have at most $n-k+1$ vertices and hence dimension $\leq n-k$.

(c) If $\alpha \subseteq \bar{D}(\sigma) - D(\sigma)$, let $\beta \subseteq sd(\mathcal{J})$ have $\hat{\sigma}$ as the last vertex and $\alpha < \beta$. Since $\alpha \notin D(\sigma)$, α has last vertex $\hat{\tau}$ for some $\tau \subseteq \mathcal{J}$ distinct from σ. From (*) we get that $\tau > \sigma$ as needed.



Note that $\bar{D}(\sigma) = \text{Cone over } \dot{D}(\sigma)$.