The epsilon-delta definition of the limit

We first define what it means for a function to have **"limit 0 at 0"**.

Definition. Let the function E be defined on an open interval about 0, except possibly at 0. We say that $\lim_{h\to 0} E(h) = 0$ if

for every challenge number $\epsilon > 0$,
there is a response number $\delta > 0$ so that
if $0 < |h| < \delta$,
then $|E(h)| < \epsilon$.

Example: $E(h) = h^2$ has limit 0 at 0:

Given an arbitrary $\epsilon > 0$,
we can choose $\delta = \sqrt{\epsilon}$. Then
if $0 < |h| < \delta$,
$|E(h)| = |h^2| < \delta^2 = \epsilon$.

Exercise: Prove these facts using epsilon-delta arguments:

- If functions E_1 and E_2 both have limit 0 at 0, so does their sum.
- If functions E_1 and E_2 both have limit 0 at 0, so does their product.
- If function E_1 has limit 0 at 0, and $|E_2(h)| < |E_1(h)|$ for all h, then E_2 has limit 0 at 0.
- If $|g(h)| < M$ all h, and E has limit 0 at 0, then the function gE has limit 0 at 0.

For a more general function F defined on an open interval about a, except possibly at a, we say

$$\lim_{x\to a} F(x) = L$$

if the "error" function $E(h) = F(a+h) - L$ has limit 0 at 0.

For example, suppose $F(x) = x^2$, and we want to show $\lim_{x\to 2} F(x) = 4$.

We let $E(h) = F(2+h) - 4 = (2+h)^2 - 4 = 4h + h^2$, and show $E(h)$ has limit 0 at 0.

If $\epsilon = \frac{1}{10}$, we can choose $\delta = \frac{1}{100}$, since then $|E(h)| = |h^2 + 4h| \le |h^2| + |4h| < \frac{1}{10000} + \frac{4}{100} < \frac{5}{100} < \frac{1}{10}$.

The case of an arbitrary ϵ is harder. Instead of finding δ directly, we could show $E_1(h) = h^2$ and $E_2(h) = 4h$ both have limit 0 at 0, and then use the fact that their sum also has limit 0 at 0.