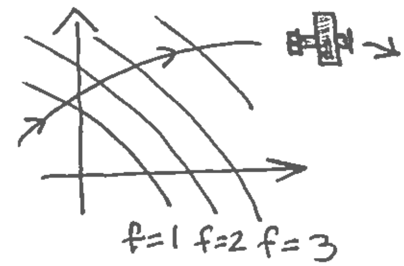
Lecture 11: The Chain Rule (§14.5) and Directional Derivatives (§14.6)

Last time: Chain Rule



① $f: \mathbb{R}^2 \to \mathbb{R}$, $x, y: \mathbb{R} \to \mathbb{R}$

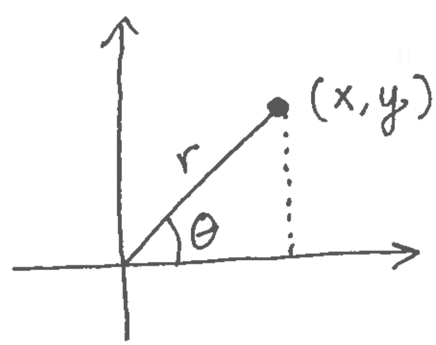
For $h(t) = f(x(t), y(t))$, we have

$$h'(t) = f_x(x(t), y(t))\, x'(t) + f_y(x(t), y(t))\, y'(t)$$

② $z = f(x,y)$ with $x = x(t)$ and $y = y(t)$. Then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

[What if x and y themselves are functions of more than one variable?]



Ex: $z = f(x,y) = x^2 + 3y$

$$x(r,\theta) = r\cos\theta$$
$$y(r,\theta) = r\sin\theta$$

So $z(r,\theta) = f(x(r,\theta), y(r,\theta)) = r^2\cos^2\theta + 3r\sin\theta$

Chain Rule: $$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}$$

$$= 2x\cdot(-r\sin\theta) + 3\cdot(r\cos\theta)$$

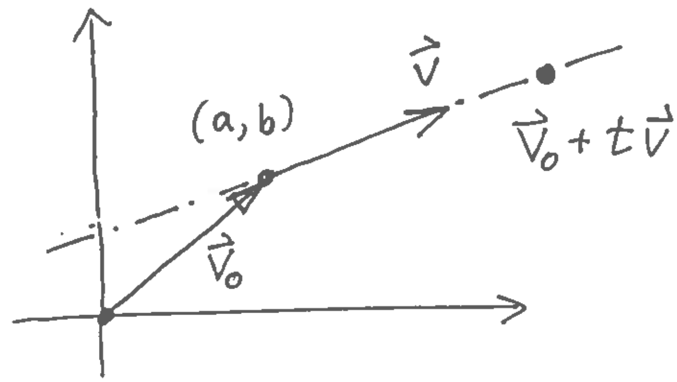
$$= (2r\cos\theta)(-r\sin\theta) + 3r\cos\theta = -2r^2\cos\theta\sin\theta + 3r\cos\theta$$

[Final comments: More vars; how to remember; other var names OK.]

Directional Derivatives (§14.6)

[Already have ∂-derivatives, measuring rates of change in x and y directions. What about other directions?
Pick a point (a,b) in $\mathbb{R}^2$ and a vector $\vec{v}$.]

The derivative of f in direction $\vec{v}$ at (a,b) is



$D_{\vec{v}} f(a,b) =$ rate of change if f as we move in direction $\vec{v}$ from (a,b).

$$= \frac{d}{dt} \underbrace{f(\vec{v}_0 + t\vec{v})}_{\text{function of one variable}} \Big|_{t=0}$$

Ex: For $\vec{v} = \vec{\imath}$ have $D_{\vec{\imath}} f(a,b) = \frac{\partial f}{\partial x}(a,b)$.

In general, compute using the Chain Rule:

Have $\vec{v}_0 + t\vec{v} = (a + tv_1, \; b + tv_2)$, so

$\vec{v}_0 = (a,b)$

$\vec{v} = (v_1, v_2)$

$f(\vec{v}_0 + t\vec{v}) = f(x,y)$ where

$x = a + tv_1$
$y = b + tv_2$

Now

$$D_{\vec{v}} f(a,b) = \frac{df}{dt}(0) = \frac{\partial f}{\partial x}(x(0), y(0))\,x'(0) + \frac{\partial f}{\partial y}(x(0), y(0))\,y'(0)$$

$$= \frac{\partial f}{\partial x}(a,b)\,v_1 + \frac{\partial f}{\partial y}(a,b)\,v_2$$

<u>Ex</u>: $f(x,y) = x^2 + y^3 \qquad \vec{u} = \frac{1}{\sqrt{2}}\vec{\imath} - \frac{1}{\sqrt{2}}\vec{\jmath}$

(Here, $\vec{u}$ is a <u>unit vector</u>; usually, we take directional derivatives in unit directions since $D_{2\vec{v}} f(a,b) = 2 D_{\vec{v}} f(a,b)$.)

$$D_{\vec{u}} f(2,1) = \frac{\partial f}{\partial x}(2,1)\cdot\frac{1}{\sqrt{2}} + \frac{\partial f}{\partial y}(2,1)\cdot\left(-\frac{1}{\sqrt{2}}\right)$$

$$= 4\cdot\frac{1}{\sqrt{2}} + 3\left(-\frac{1}{\sqrt{2}}\right) = 1/\sqrt{2} \approx 0.7071$$

<u>Gradient</u>: For $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$, define

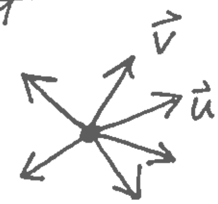
$$\nabla f(a,b) = \left(\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b)\right)$$

<u>Ex</u>: For $f = x^2 + y^3$, $\nabla f(2,1) = (4,3)$

[Will explain the geometric meaning shortly, but for now notice:]

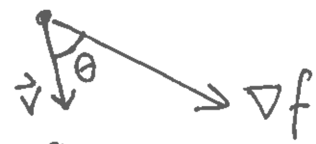
$$D_{\vec{v}} f(a,b) = \nabla f(a,b) \cdot \vec{v}$$

<u>Q</u>: In what direction does f increase fastest at (a,b)?



[why might we need to know this?]

Suppose $\vec{v}$ is a unit vector. Then



$$D_{\vec{v}} f(a,b) = \nabla f(a,b) \cdot \vec{v} = |\nabla f(a,b)| \cos\theta$$

To maximize, want $\theta = 0$, that is

$$\vec{v}_{max} = \frac{\nabla f(a,b)}{|\nabla f(a,b)|}.$$

Note also that

$$D_{\vec{v}_{max}} f(a,b) = |\nabla f(a,b)|$$

<u>Summary</u>: $\nabla f(a,b)$ points in the direction that f increases fastest at (a,b). Its length is the rate of said increase.

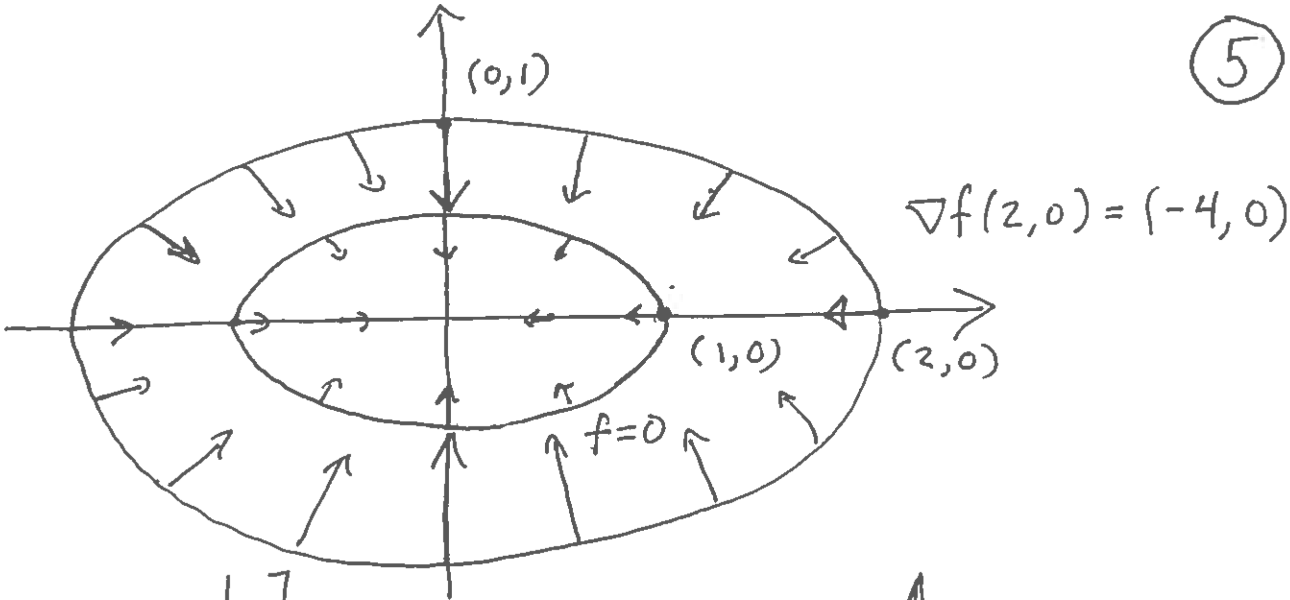
<u>Ex</u>: $f(x,y) = 1 - x^2 - 4y^2$ $\quad \nabla f = (-2x, -8y)$

<u>Level sets</u>: $f = 0$: $1 - x^2 - 4y^2 = 0 \iff x^2 + 4y^2 = 1$

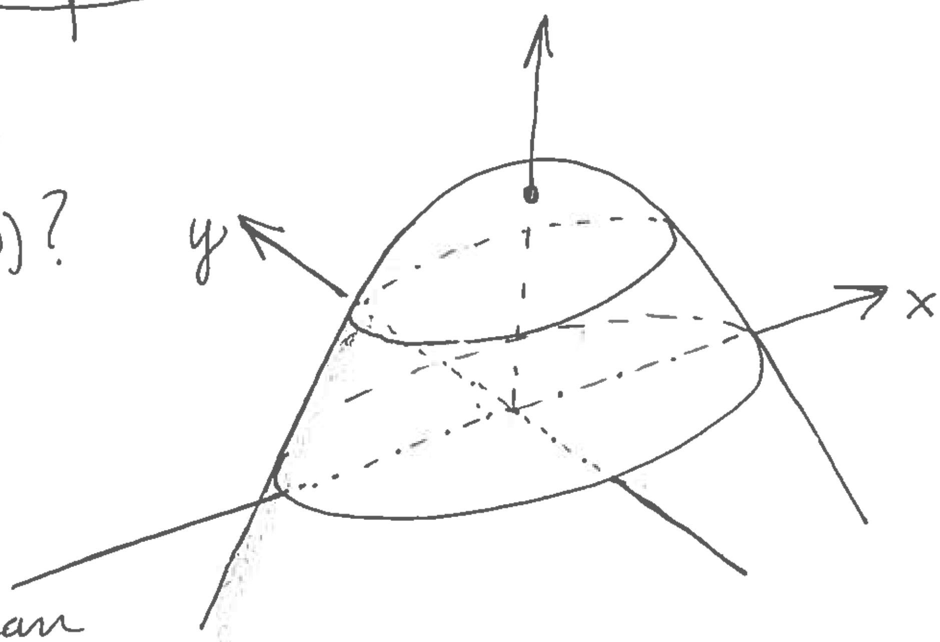
$f = -3$: $1 - x^2 - 4y^2 = -3 \iff x^2 + 4y^2 = 4$

$$\iff \frac{x^2}{4} + y^2 = 1.$$

<u>Gradient</u>: $\nabla f(1,0) = (-2,0)$ $\quad \nabla f(0,1) = (0,-8)$



[Go to visualization!]



<u>Q</u>: What is $\nabla f(0,0)$?

<u>A</u>. $\vec{0}$.

<u>Morals</u>:

- A min/max can only occur when $\nabla f = \vec{0}$.
- ∇f is always at right angles to the level sets.