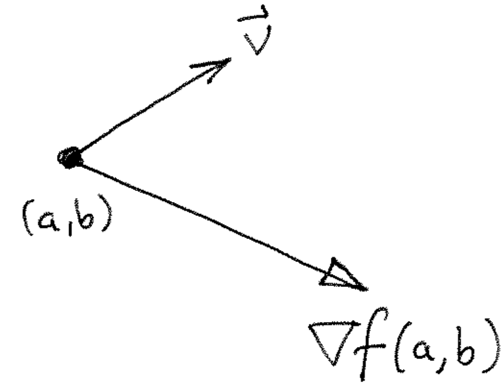
Lecture 12: More on the gradient (§14.6); overview of optimization (§14.7 – 14.8)

Last time: $f : \mathbb{R}^2 \to \mathbb{R}$



$D_{\vec{v}} f(a,b) =$ Rate f changes as move in direction $\vec{v}$ starting at (a,b).

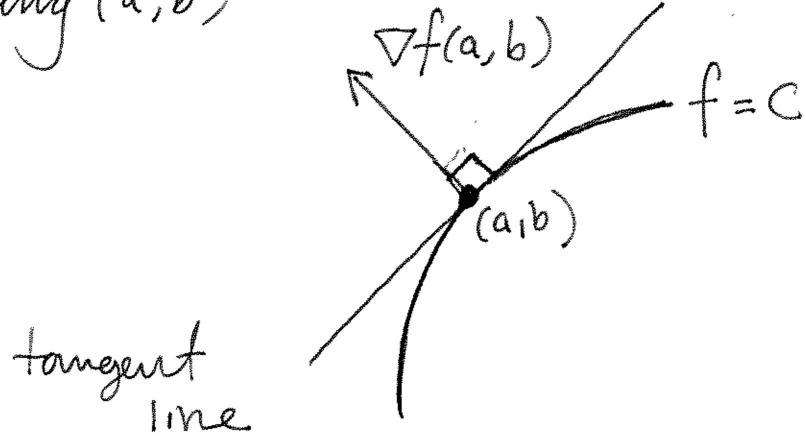
$$\nabla f(a,b) = \left(\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b) \right)$$

Relationship: $D_{\vec{v}} f(a,b) = \nabla f(a,b) \cdot \vec{v}$

Meaning: $\nabla f(a,b)$ points in the direction of fastest increase of f at (a,b). $|\nabla f(a,b)|$ is the rate of that increase.

Key Props: ① ∇f is at right angles to level sets.
② $\nabla f = \vec{0}$ at a local min/max.

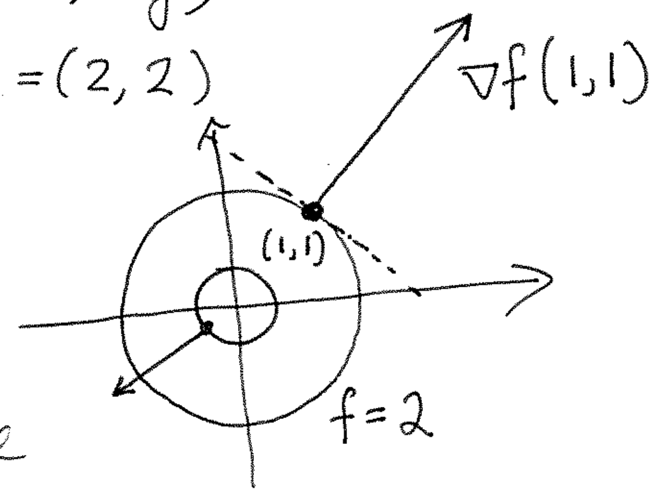
① $\nabla f(a,b)$ is perpendicular to the level set of f containing (a,b)



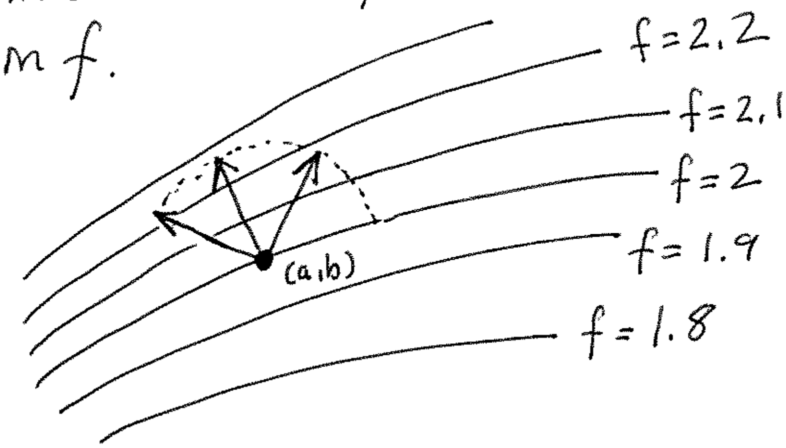
Ex: $f(x,y) = x^2 + y^2$ $\quad \nabla f = (2x, 2y)$

$f(1,1) = 2$ $\quad \nabla f(1,1) = (2,2)$

Level set $f=2$: $x^2+y^2=2$



Reason 1: Gradient points in direction of fastest increase in f.

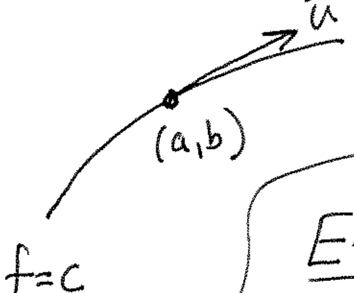


To cross as many level sets as possible with a vector of fixed length, go at right angles to level sets.

Reason 2: Suppose $\vec{u}$ is tangent to the level set of f at (a,b). Since f is constant on the level set,

$$0 = D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u}$$

Thus ∇f and $\vec{u}$ are $\perp$.



Ex: Find the tangent line to the ellipse C given by $5x^2 + 5y^2 - 6xy = 16$.

Solution: View C as the level set $f = 16$ for

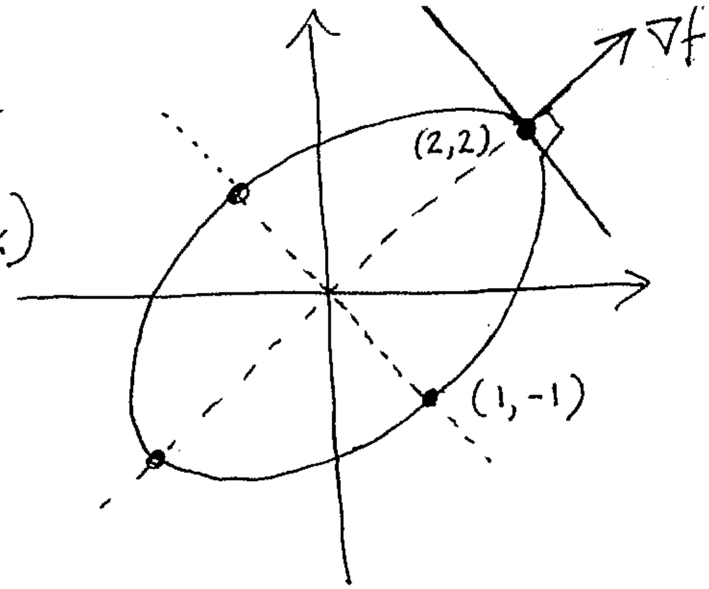
$f(x,y) = 5x^2 + 5y^2 - 6xy$

So $\nabla f = (10x - 6y, 10y - 6x)$

and $\nabla f(2,2) = (8,8)$

Thus the tangent line is

$x + y = 4$.



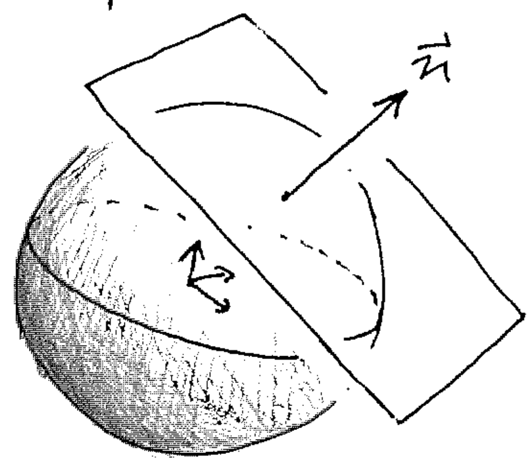
Q: Find the tangent plane to the sphere

$$x^2 + y^2 + z^2 = 6 \quad \text{at } (1,1,2)$$

A: Take $f(x,y,z) = x^2 + y^2 + z^2$. Then $\nabla f = (2x, 2y, 2z)$, and so as ∇f is $\perp$ the level set (= sphere) we use

$\vec{n} = \nabla f(1,1,2) = (2,2,4)$

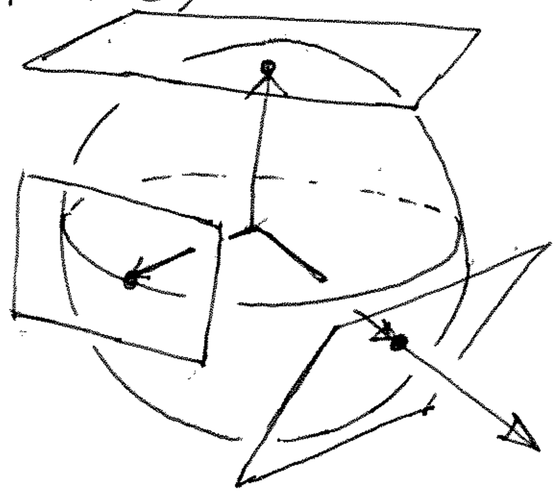
as the normal to our tangent plane. So the eqn is



$$2(x-1) + 2(y-1) + 4(z-2) = 0$$

$$\iff x + y + 2z = 6$$

<u>General Fact</u>: If S is a sphere centered at $O=(0,0,0)$ and P is a point in S, then $\vec{a}=\overrightarrow{OP}$ is a normal vector to the tangent plane to S at P



<u>Reason</u>: $f(x,y,z)=x^2+y^2+z^2$

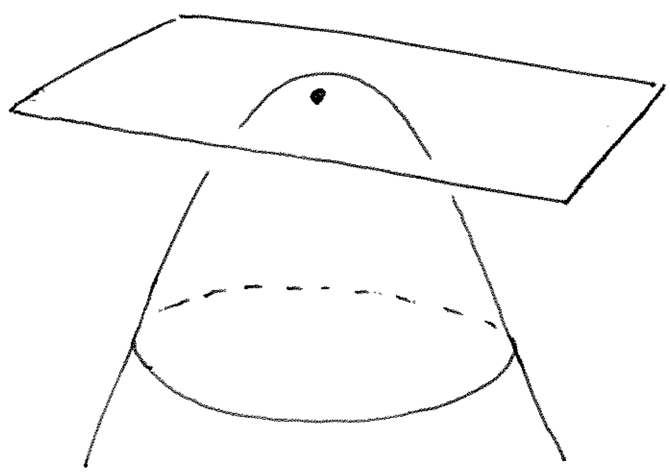
$$\nabla f=(2x,2y,2z)$$

<u>Optimization</u>: For $f:\mathbb{R}^2\to\mathbb{R}$, want to find (a,b) where it is largest (or smallest).

Suppose f is largest at (a,b). Then $\nabla f(a,b)=\vec{0}$ [since can't increase f, after all.]

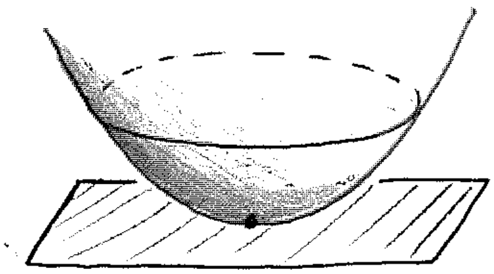
<u>Ex</u>: $f(x,y)=-x^2-y^2$

$$\nabla f=(-2x,-2y)$$



f is largest at $(0,0)$ and $\nabla f(0,0)=(0,0)$

Suppose instead $f(a,b)$ is the minimum value of f. What is $\nabla f(a,b)$?



Ex: $f(x,y) = x^2+y^2$ has a min at $(0,0)$ where $\nabla f = (2x, 2y)$ is $\vec{0}$

[Even though no matter which direction we go, f increases ?!]

In general, $\nabla f(a,b) = \vec{0}$ when $f(a,b)$ is minimal as well.

Note: $\nabla f(a,b) = \vec{0} \iff$ tangent plane is parallel to the xy-plane.

Reason: The general tangent plane eqn is:

$$z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

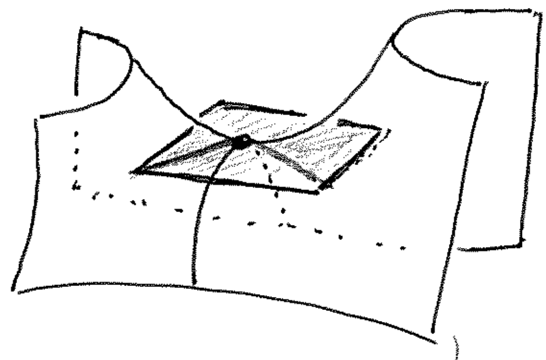
So if $\nabla f(a,b) = \vec{0}$ this becomes $z - f(a,b) = 0$.

Another example where $\nabla f = \vec{0}$:

$$f(x,y) = x^2 - y^2$$

$$\nabla f = (2x, -2y)$$

$$\nabla f(0,0) = \vec{0}$$



Q: How can we tell these apart? Need a 2^{nd} derivative test:

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \underbrace{\frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial^2 f}{\partial y \partial x}}$$

Typically equal by Clairaut's Thm, e.g. if both are continuous.

[Saw this in section.]