

Lecture 14: More on min/max (§14.7)

①

Last time: $f: \mathbb{R}^2 \rightarrow \mathbb{R}$. Critical point: $\nabla f(a,b) = \vec{0}$

$$D = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{vmatrix}$$

If $D > 0$ and

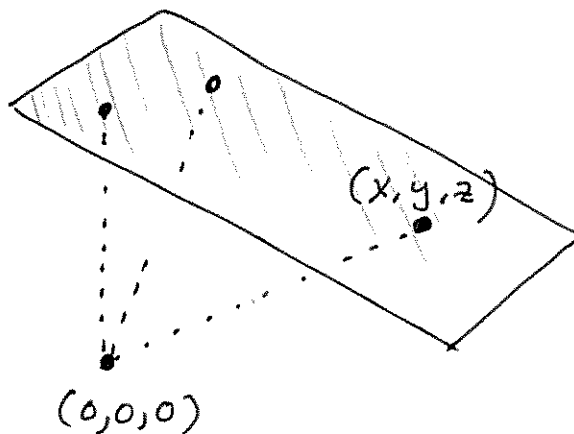
$f_{xx}(a,b) > 0 \Rightarrow$ local min

$f_{xx}(a,b) < 0 \Rightarrow$ local max

If $D < 0 \Rightarrow$ saddle

Ex: Find the distance from $(0,0,0)$ to the plane
 $x - y + 2z = 6$.

$$z = \frac{6 - x + y}{2}$$



Need to minimize:

$$f(x,y) = \left(\begin{array}{l} \text{distance from} \\ (x,y, \frac{6-x+y}{2}) \\ \text{to } (0,0,0) \end{array} \right)^2 = x^2 + y^2 + \frac{1}{4}(6-x+y)^2$$

Critical Points: $\nabla f = \vec{0}$

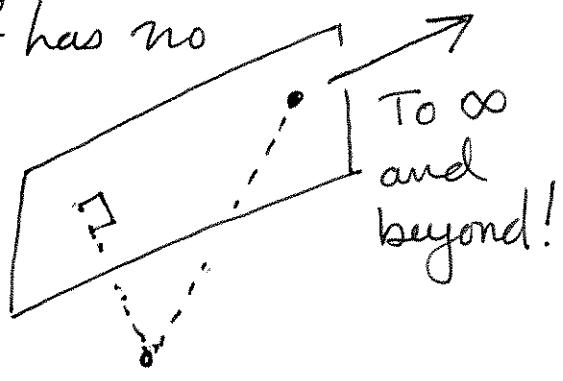
$$\frac{\partial f}{\partial x} = 2x + \frac{1}{2}(6-x+y) \cdot (-1) = \frac{5}{2}x - \frac{1}{2}y - 3$$

$$\frac{\partial f}{\partial y} = 2y + \frac{1}{2}(6-x+y) = -\frac{1}{2}x + \frac{5}{2}y + 3$$

(2)

Setting $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$, we find a unique solution $x=1$ and $y=-1$. Since this is the only critical point, the minimum distance must be $\sqrt{f(1,-1)} = \sqrt{1^2 + 1^2 + \frac{1}{4}4^2} = \sqrt{6}$.

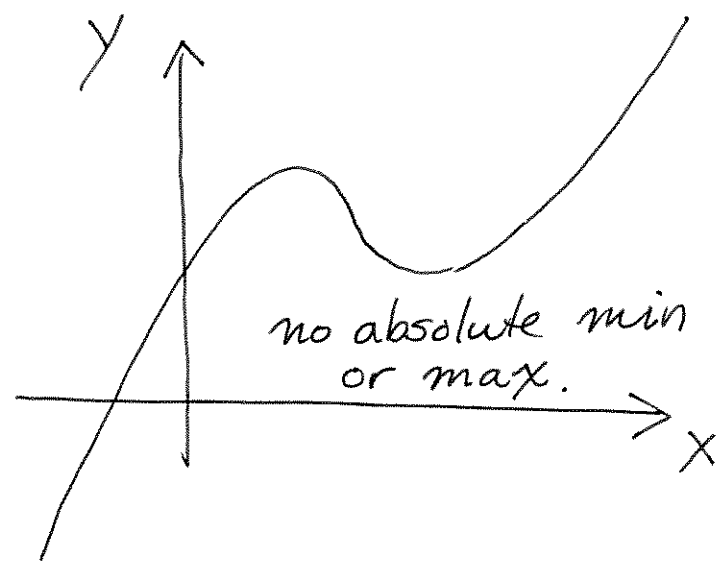
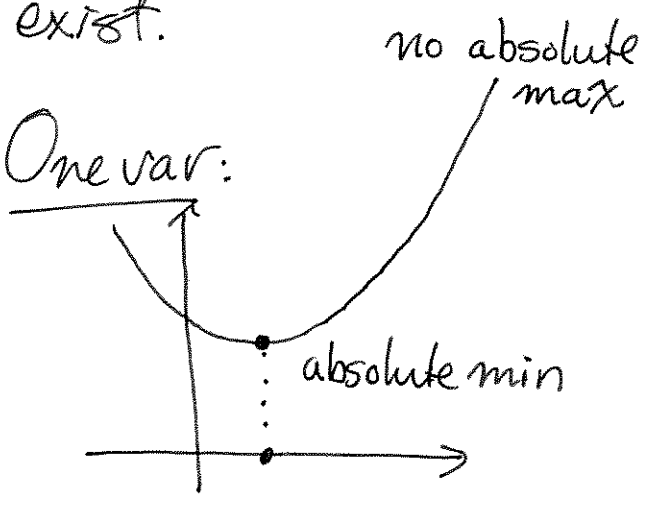
Hey, the same reasoning says that f takes its maximum at $(1,-1)$?! In fact, f has no maximum as is clear geometrically and also algebraically:



$$f(x,0) = x^2 + \frac{1}{4}(-x)^2 = \frac{5}{4}x^2$$

For this problem, we know there should be a min for geometric reasons, but let's look at some general criteria for absolute min/max to exist.

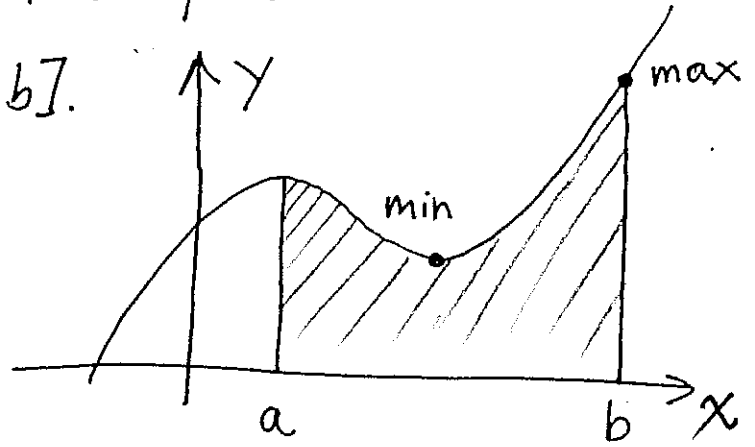
One var:



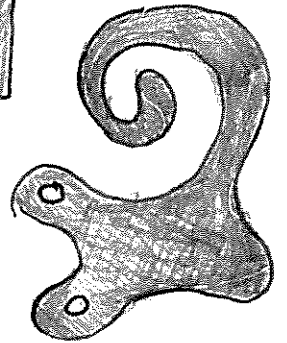
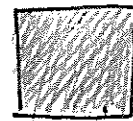
Extreme Value Theorem: Suppose f is continuous on $[a, b] = \{a \leq x \leq b\}$. Then f has both an absolute min and max on $[a, b]$. ③

These occur at either

- a) a critical point of f
- b) one of a or b .

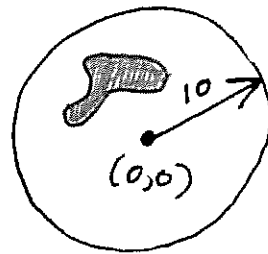


Two Var: D subset of \mathbb{R}^2 :



Bounded: Contained within a disk:

Closed: Contains all its boundary points.



Closed:

$x^2 + y^2 \leq 1$

Not Closed:

$x^2 + y^2 < 1$

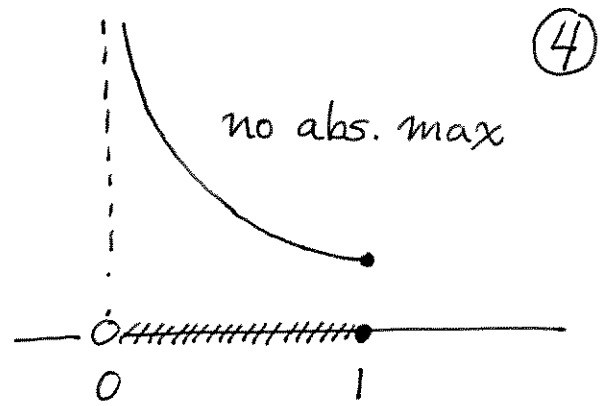
$x^2 + y^2$

$\begin{cases} 0 < x \leq 1 \\ 0 \leq y \leq 1 \end{cases}$

$\begin{cases} 0 \leq x \leq 1 \\ 0 \leq y \leq 1 \end{cases}$

Closed important in 1-var too

$$f = \frac{1}{x} \text{ on } (0, 1] = \{0 < x \leq 1\}$$



Extreme Value Theorem:

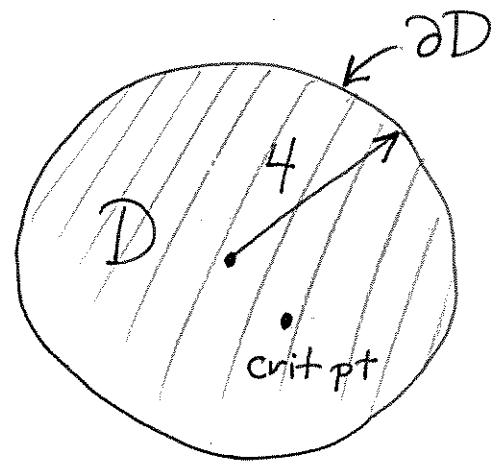
Suppose f is continuous on D in \mathbb{R}^n . If D is closed and bounded, then f has absolute min and max on D . These occur either at critical points of f or on the boundary of D .

Back to original problem:

$f(x, y) = x^2 + y^2 + \frac{1}{4}(6 - x + y)^2$ on \mathbb{R}^2
which has one critical point at $(1, -1)$.

Consider $D = \{x^2 + y^2 \leq 16\}$

On D , f has one crit pt
and there $f = 6$.



On ∂D , $f(x, y) \geq x^2 + y^2 = 16$. So 6 is
the absolute min value of f on D .

Note: $f(x, y) \geq x^2 + y^2 \geq 16$ outside D ,
so 6 is the absolute min of f on all of \mathbb{R}^2 .

(5)

Asides: 1) Turns out max on D is $33 + 12\sqrt{2} \approx 50$
and occurs at $(-2\sqrt{2}, 2\sqrt{2})$; will learn
how to figure this out next time.

2) Good Check: $f_{xx} = 5/2$, $f_{xy} = -1/2$, $f_{yy} = 5/2$

$$D = \begin{vmatrix} 5/2 & -1/2 \\ -1/2 & 5/2 \end{vmatrix} = \frac{25}{4} - \frac{1}{4} = 6 > 0 \text{ and } f_{xx} > 0.$$

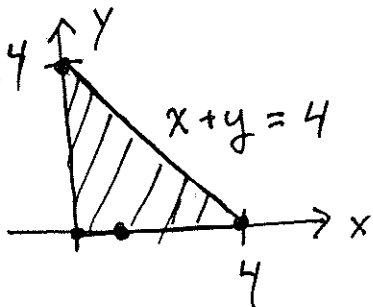
So the 2nd derivative test says there is a
local min at $(1, -1)$.

3) There are weird functions like $x^2 + y^2(1-x)^3$
which have only one critical point, a local
min, but no absolute min!

Ex: $f = y - xy$ $D = \{x, y \geq 0, x+y \leq 4\}$

(6)

$\nabla f = (-y, 1-x)$



So a unique crit. pt at $(1,0)$. By E.V.T, f has absolute min/max on D , both of which must occur on ∂D as no crit. pts inside D .

a) Along the bottom: $f(x,0) = 0$ for any x .

b) Along the right side: $f(0,y) = y$, so min on this side is 0, max is 4.

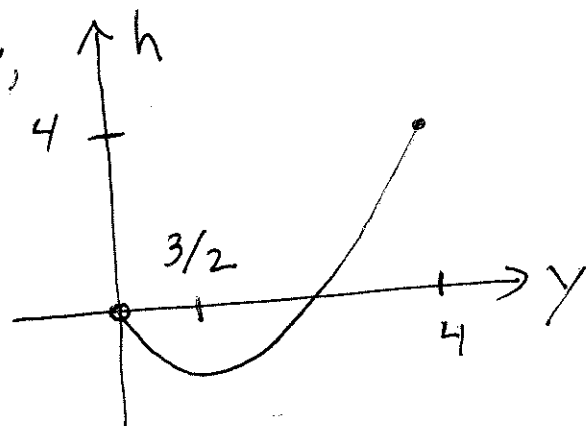
c) Along the sloped side we have

$$f(4-y, y) = y - (4-y)y = y^2 - 3y = h(y)$$

Note h has one crit pt, namely,

$$y = 3/2, \text{ since } h'(y) = 2y - 3$$

$$\text{where } h = -\frac{9}{4} = -2\frac{1}{4}.$$



So the absolute max of f on D is 4, occurring at $(0,4)$ and the absolute min is $-9/4$ occurring at $(5/2, 3/2)$.