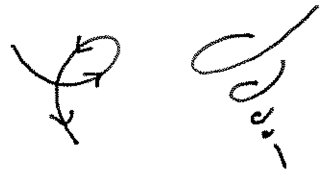
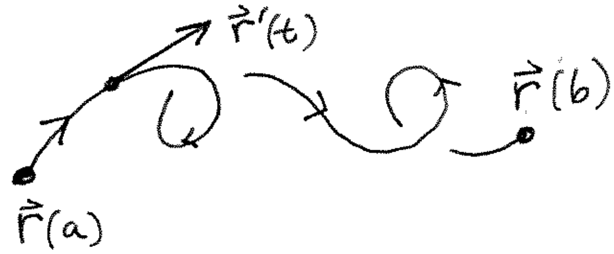
Lecture 17: Integration along curves (§13.3 and 16.2)

Last time: $\vec{r}: \mathbb{R} \to \mathbb{R}^2$ or $\mathbb{R}^3$

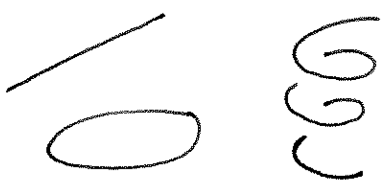


$$\text{Length} = \int_a^b |\vec{r}\,'(t)|\, dt$$



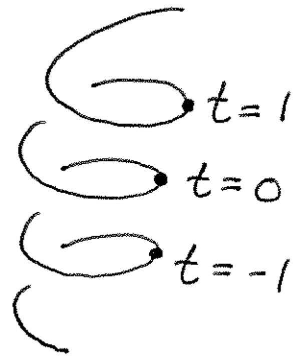
Really two concepts:

Curve: A set of points in $\mathbb{R}^3$ looking like:

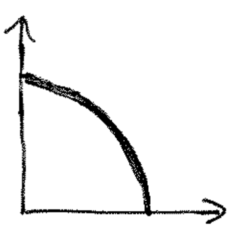


Parameterization: $\vec{r}: \mathbb{R} \to \mathbb{R}^3$

Instructions for moving along a curve.



[Any curve has many parameterizations.]

Ex:  $C = 1/4$ of a unit circle in $\mathbb{R}^2$.

ⓐ $\vec{r}: [0, \pi/2] \to \mathbb{R}^2$ $\quad \vec{r}(t) = (\cos t, \sin t)$

[Parameterizing by angle]

$$\vec{r}\,'(t) = (-\sin t, \cos t)$$

$$|\vec{r}\,'(t)| = \sqrt{(-\sin t)^2 + \cos^2 t} = 1$$

$$\text{Length} = \int_0^{\pi/2} |\vec{r}\,'(t)|\, dt = \int_0^{\pi/2} 1\, dt = \pi/2.$$

(b) $\vec{r}:[0,1]\longrightarrow \mathbb{R}$ $\quad \vec{r}(t)=(t,\sqrt{1-t^2})$

[Parameterizing by x]

$$\vec{r}\,'(t)=\left(1,\tfrac{1}{2}(1-t^2)^{-1/2}\cdot(-2t)\right)$$
$$=\left(1,-t/\sqrt{1-t^2}\right)$$
$$|\vec{r}\,'(t)|=\sqrt{1^2+\frac{t^2}{1-t^2}}=\frac{1}{\sqrt{1-t^2}}$$

$$\text{Length}=\int_0^1\frac{1}{\sqrt{1-t^2}}\,dt=\arcsin(t)\Big|_{t=0}^{1}=\pi/2-0=\boxed{\pi/2}\ \checkmark$$

Integrating a function on a curve (§ 16.2)

C curve in $\mathbb{R}^2$ $\qquad f:\mathbb{R}^2\longrightarrow\mathbb{R}$

$$\int_C f\,ds=\int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$$

where $\vec{r}:[a,b]\longrightarrow\mathbb{R}^2$ is a parameterization of C.

[Turns out not to depend on $\vec{r}$, just C.]

Some meanings:

① f = temperature

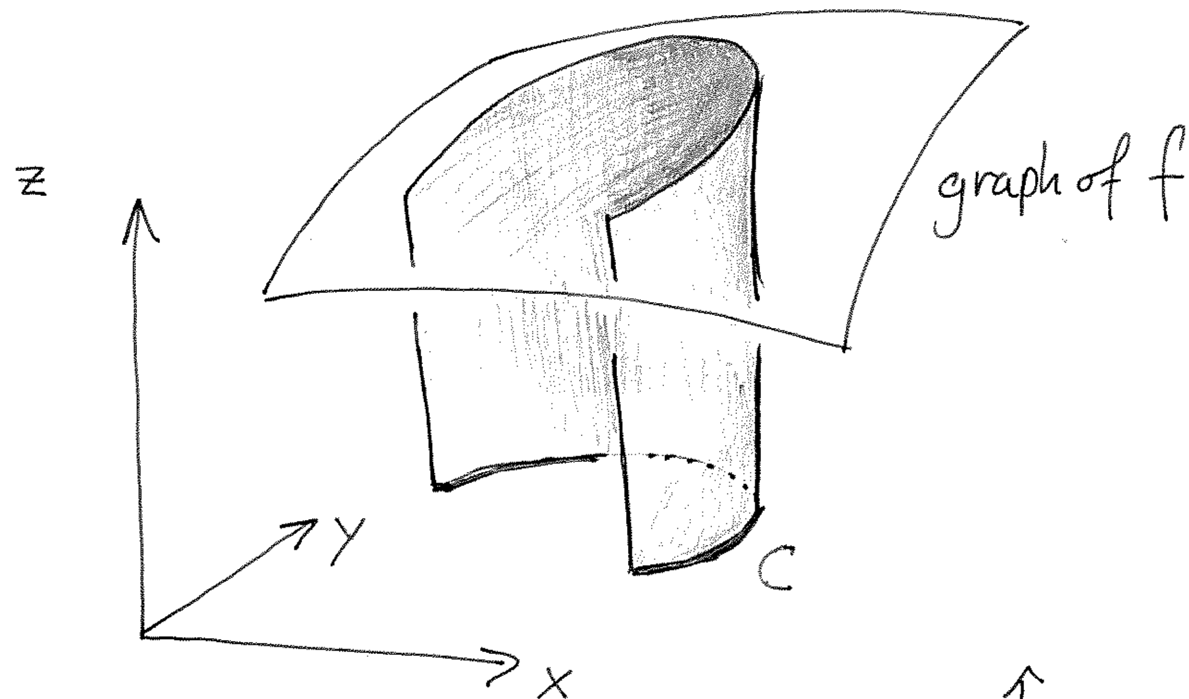
Compare: Average of $f:\mathbb{R}\to\mathbb{R}$ on $[a,b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$

$$\text{Average temperature along } C=\frac{1}{\text{Length}(C)}\int_C f\,ds$$

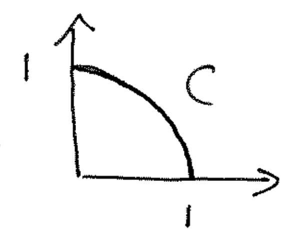
2. f = density of material curve is made of $\left(\text{mass}/\text{length}\right)$

$$\text{Mass of curve} = \int_C f\,ds$$

3. Area of region above C and below the graph of f.



Ex: Find the average of $f(x,y) = x$ on



$\vec{r}(t) = (\cos t, \sin t) \quad 0 \le t \le \pi/2$

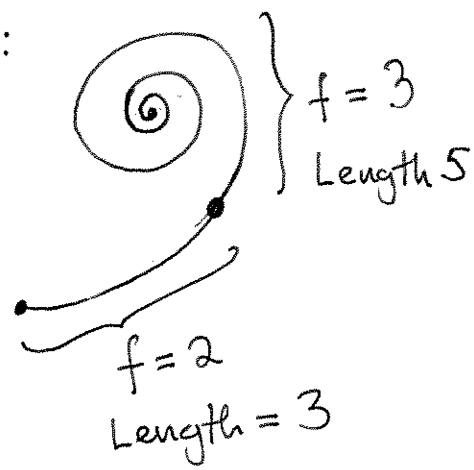
$$\int_C f\,ds = \int_0^{\pi/2} f(\vec{r}(t))\,|\vec{r}\,'(t)|\,ds = \int_0^{\pi/2} \cos t \cdot 1\,dt$$

$$= \sin t \Big|_0^{\pi/2} = \sin(\pi/2) - \sin(0) = 1 - 0 = 1$$

$$\text{Average} = \frac{1}{\text{Length}} \int_C f\,ds = \frac{2}{\pi} \approx 0.6366$$

Understanding these integrals:

Ex:



[What is the averge here? First guess of 2.5 is too small...]

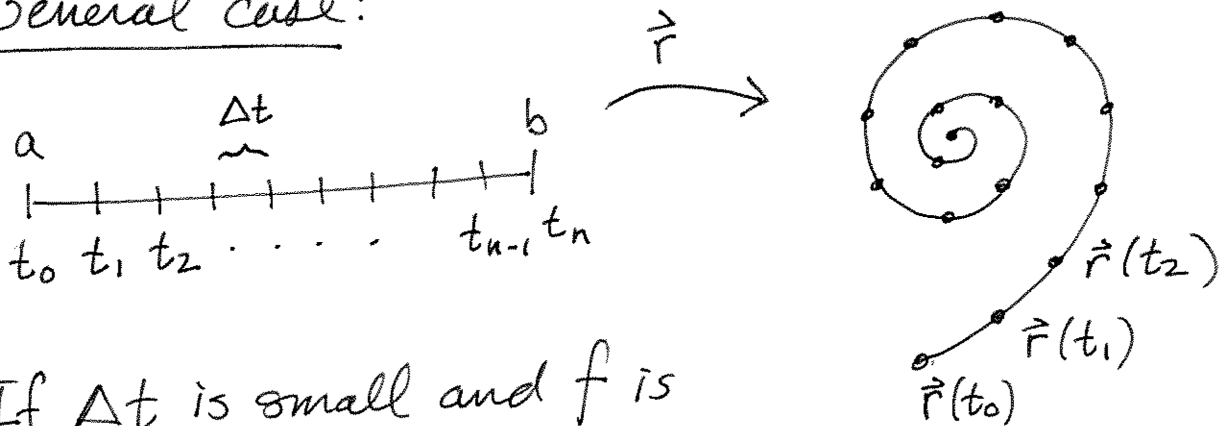
Average of f on C

$$= \begin{pmatrix}\text{portion}\\ \text{where}\\ f=2\end{pmatrix}\cdot 2 + \begin{pmatrix}\text{portion}\\ \text{where}\\ f=3\end{pmatrix}\cdot 3$$

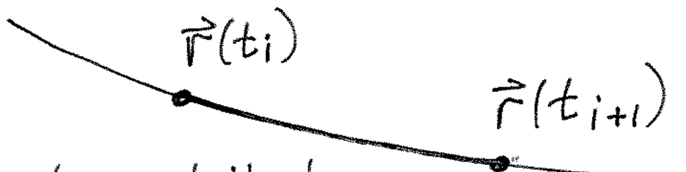
$$= \left(\frac{3}{8}\right)\cdot 2 + \left(5/8\right)\cdot 3$$

$$= 21/8 = 2\,5/8$$

General case:



If Δt is small and f is continuous, then f is almost constant on



So this segment contributes

$$\approx \frac{(\text{length of seg.})}{(\text{length of } C)}\, f(\vec{r}(t_i))$$

to the average.

Now the segment has length $\approx |\vec{r}\,'(t_i)|\,\Delta t$ by linear approximation, and so

$$\text{Average} \approx \sum_{i=0}^{n-1} \frac{(\text{len of } i^{th} \text{ seg})}{(\text{len of } C)} f(\vec{r}(t_i))$$

$$\approx \frac{1}{\text{len}(C)} \sum_{i=0}^{n-1} f(\vec{r}(t_i))\, |\vec{r}\,'(t_i)|\, \Delta t$$

As $\Delta t \to 0$, we get

$$\text{Average} = \frac{1}{\text{len}(C)} \int_a^b f(\vec{r}(t))\, \underbrace{|\vec{r}\,'(t)|\, dt}$$

$$= \frac{1}{\text{len}(C)} \int_C f\, ds \leftarrow$$

Arc length element

<u>Compare:</u>

$$\text{Average of } f \text{ on } [a,b] = \frac{1}{b-a} \int_a^b f(t)\, dt$$