


Lecture 18: Vector fields (§16.1 and 16.2)

①

Last time:

 C curve in \mathbb{R}^2
 $f: \mathbb{R}^2 \rightarrow \mathbb{R}$


$$\int_C f ds = \int_a^b f(\vec{r}(t)) |\vec{r}'(t)| dt$$

where $\vec{r}: [a, b] \rightarrow \mathbb{R}^2$ is a parameterization of C

Uses:

① Average of f on $C = \frac{1}{\text{Len}(C)} \int_C f ds$

② Total mass = $\int_C f ds$ ← density fn.

③ Area  graph of f

Note: $ds = |\vec{r}'(t)| dt$ is the arc-length element

Also $\int_C 1 ds = \text{Length}(C)$

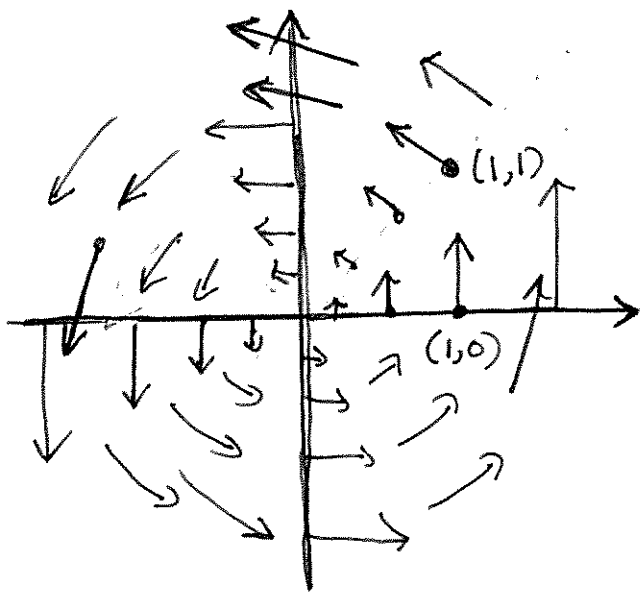
[Everything works for curves in \mathbb{R}^3 except use ③.
Will learn more about diff param. of the same C in section.]

Vector fields (§16.1 and 16.2)

For \mathbb{R}^2 , a vector field is a function $\vec{F}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

Ex: $\vec{F}(x, y) = -\frac{y}{2} \vec{i} + \frac{x}{2} \vec{j}$

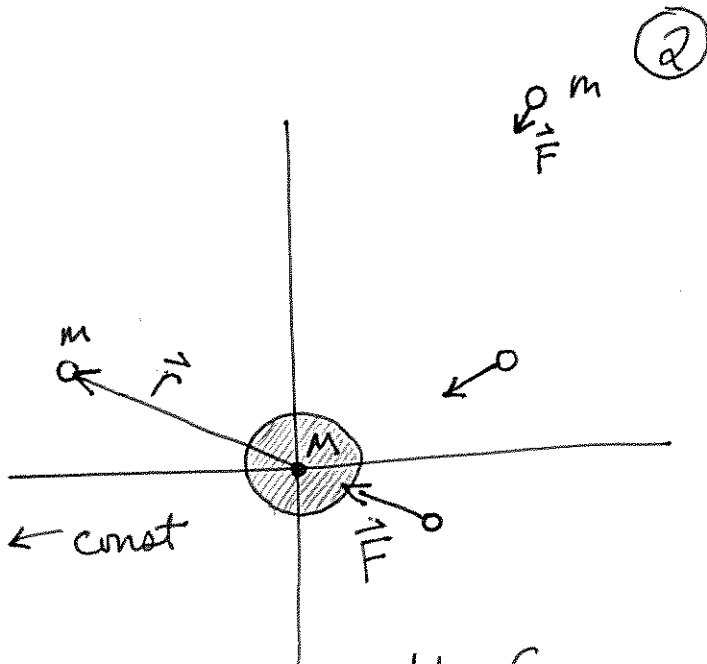
- Uses:
- Wind speed/direction
 - Fluid flow
 - Force magnitude/direction
 - Electric/magnetic fields



Ex: Gravity

Large mass M at $(0,0)$.

Force \vec{F} on small mass m depends on position $\vec{r} = (x,y)$ and points in direction $-\vec{r}$.



Newton's Law: $|\vec{F}| = \frac{MmG}{|\vec{r}|^2}$ ← const

As $\vec{F} = -C\vec{r}$, have $|\vec{F}| = C|\vec{r}| \Rightarrow C = \frac{MmG}{|\vec{r}|^3}$

$$\text{So } \vec{F} = -\frac{MmG}{|\vec{r}|^3} \vec{r}$$

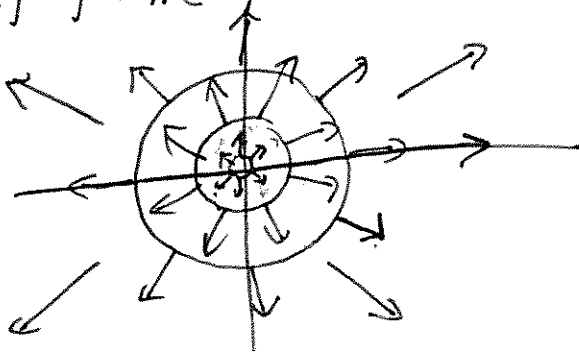
[For several bodies, add vector fields. An electric field generated by a charged particle is similar.]

[Where have we seen vector fields before in this class?]

Ex: Gradients: If $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ then $\nabla f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

① $f(x,y) = x^2 + y^2$

$$\nabla f = (2x, 2y)$$



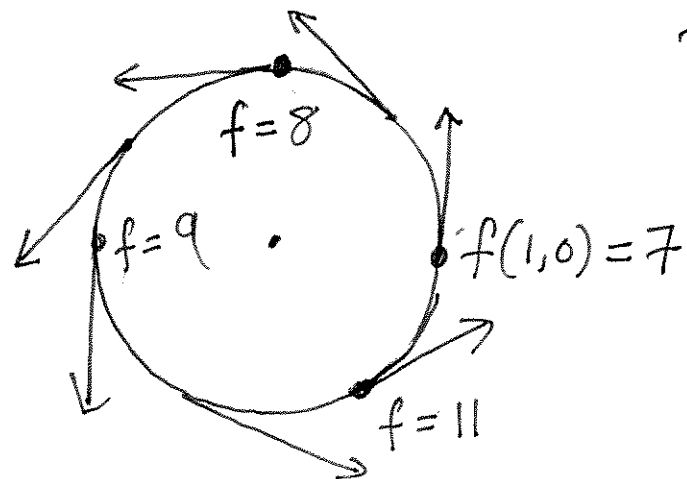
② $f(x,y) = \frac{MmG}{\sqrt{x^2 + y^2}} = \frac{MmG}{|\vec{r}|}$. Then

$$\begin{aligned} \nabla f &= MmG \left(-\frac{1}{2}(x^2+y^2)^{-3/2} \cdot (2x), \quad \text{---} \right) \quad (3) \\ &= -MmG \left(\frac{x}{(x^2+y^2)^{3/2}}, \frac{y}{(x^2+y^2)^{3/2}} \right) \\ &= \frac{-MmG}{|\vec{r}|^3} \vec{r} = \vec{F} \end{aligned}$$

In general, when $\vec{F} = \nabla f$ we call f a potential function for \vec{F} and say \vec{F} is a conservative vector field. [Think potential energy / conservation of energy.]

Q: Is $\vec{F} = -\frac{y}{2}\vec{i} + \frac{x}{2}\vec{j}$ conservative?

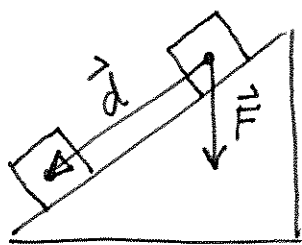
A: No. Suppose $\vec{F} = \nabla f$. Since \vec{F} is tangent to the unit circle, following the circle increases f .



But going all the way around, we end up back at $(1,0)$.

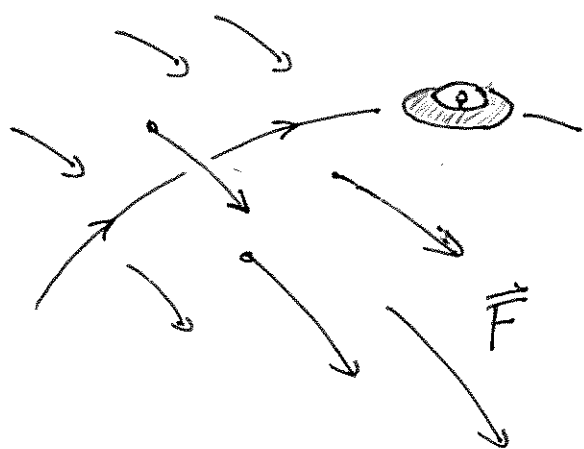
Integrating Vector Fields (§16.2)

(4)



Work done by gravity: $W = \vec{F} \cdot \vec{d}$
[Assumes const force.]

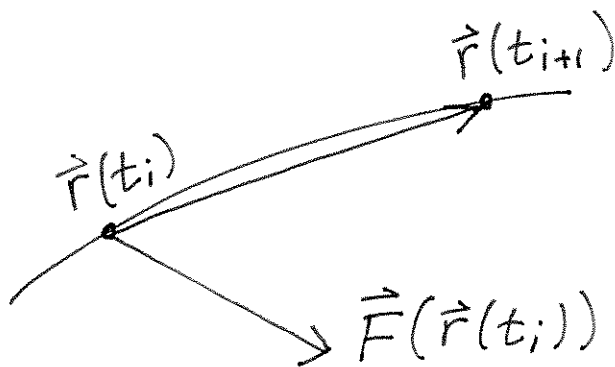
How much work does gravity do here?



Motion of ship $\vec{r}: \mathbb{R} \rightarrow \mathbb{R}^2$

Force of gravity: $\vec{F}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

Break into segments:



$$\begin{aligned} \text{Work done here} &\approx \vec{F}(\vec{r}(t_i)) \cdot (\vec{r}(t_{i+1}) - \vec{r}(t_i)) \\ &\approx \vec{F}(\vec{r}(t_i)) \cdot \Delta t \vec{r}'(t_i) \\ &\approx (\vec{F}(\vec{r}(t_i)) \cdot \vec{r}'(t_i)) \Delta t \end{aligned}$$

Sum up and take $\Delta t \rightarrow 0$ to learn

$$\text{Total Work} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$$

General Setup: C curve in \mathbb{R}^n

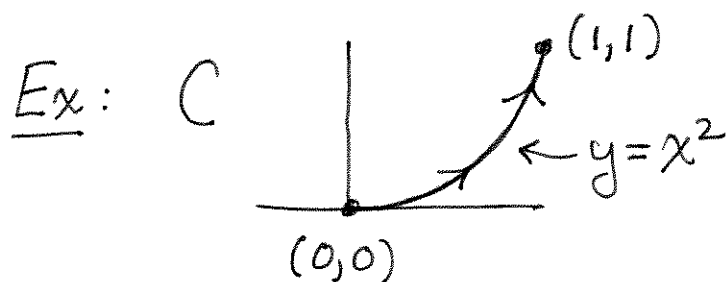
(5)

Set $\vec{F}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ a vector field

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$$

for any parameterization $\vec{r}: [a, b] \rightarrow \mathbb{R}^n$ of C .

[Note: Answer only depends on direction of param.]



$$\vec{r}(t) = (t, t^2) \text{ for } 0 \leq t \leq 1$$

$$\vec{r}'(t) = (1, 2t)$$

For $\vec{F} = (y, 1)$ we have

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt = \int_0^1 (t^2, 1) \cdot (1, 2t) dt$$

$$= \int_0^1 t^2 + 2t dt = \left. \frac{t^3}{3} + t^2 \right|_{t=0}^{t=1} = \frac{4}{3}$$

Here's why this is consistent with the work interpretation of $\int_C \vec{F} \cdot d\vec{r}$

