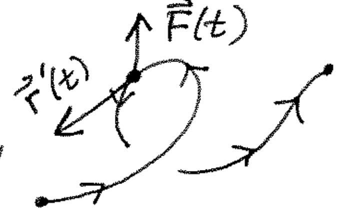
Lecture 19: More on integrating vector fields (§16.2)

<u>Last time</u>: C oriented curve in $\mathbb{R}^n$

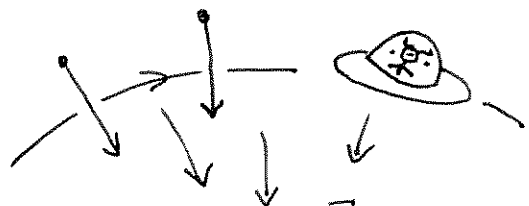
$\vec{F}: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ vector field



$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$$

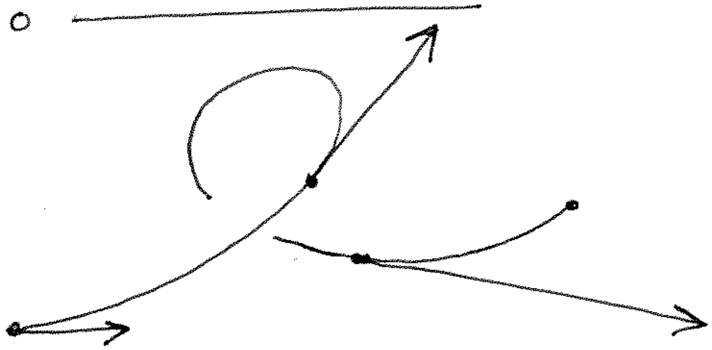
for any parameterization $\vec{r}: [a,b] \longrightarrow \mathbb{R}^n$ of C.

<u>Ex</u>: Work $= \int_C \vec{F} \cdot d\vec{r}$



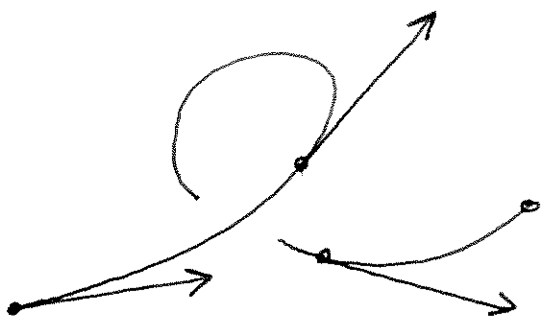
[Other examples from E+M, see HW.]

Alternate viewpoint.

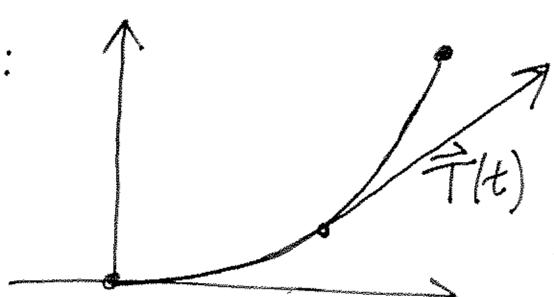


Unit tangent vectors

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$



<u>Ex</u>:



$\vec{r}(t) = (t, t^2)$ $\quad \vec{r}'(t) = (1, 2t)$

$|\vec{r}'(t)| = \sqrt{1+4t^2}$

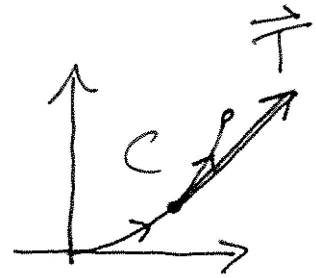
$$\vec{T}(t) = \left(\frac{1}{\sqrt{1+4t^2}}, \frac{2t}{\sqrt{1+4t^2}} \right)$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\, dt \qquad \vec{r}\,'(t) = |\vec{r}\,'(t)|\,\vec{T}(t)$$

$$= \int_a^b \left(\vec{F}(\vec{r}(t))\cdot \vec{T}(t)\right) \underbrace{|\vec{r}\,'(t)|\, dt}_{ds}$$

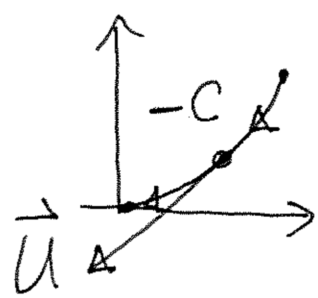
$$= \int_C \vec{F}\cdot\vec{T}\, ds$$ [Integral of a fn with respect to arc length]

Note that ↑ doesn't change if we use a different parameterization, unless we travel the other way along C and then $\vec{T}(t) \longleftrightarrow -\vec{T}(t)$.



Thus if $-C = C$ oriented the other way have:

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\vec{T}\, ds$$



Integrals of functions don't depend on orient

$$= \int_{-C} \vec{F}\cdot\vec{T}\, ds$$

$$= -\int_{-C} \vec{F}\cdot\vec{U}\, ds = -\int_{-C} \vec{F}\cdot d\vec{r}$$

[So switching orientation of C changes the sign of $\int_C \vec{F}\cdot d\vec{r}$]

<u>Alterate notation</u>: C curve in $\mathbb{R}^2$, parameterized by $\vec{r}:[a,b] \longrightarrow \mathbb{R}^2$, and $\vec{F}:\mathbb{R}^2 \to \mathbb{R}^2$ a vector field

$$\vec{r}(t) = (x(t),\ y(t)) = x(t)\,\vec{\imath} + y(t)\,\vec{\jmath}$$

$$\vec{F}(x,y) = P(x,y)\,\vec{\imath} + Q(x,y)\,\vec{\jmath}$$

Then

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \overbrace{\vec{F}(\vec{r}(t))}^{P(\vec{r}(t))\vec{\imath} + Q(\vec{r}(t))\vec{\jmath}} \cdot \underbrace{\vec{r}\,'(t)}_{x'(t)\vec{\imath} + y'(t)\vec{\jmath}}\, dt$$

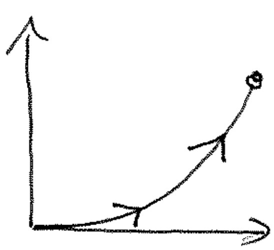
$$= \int_a^b \Big(P(\vec{r}(t))\,x'(t) + Q(\vec{r}(t))\,y'(t) \Big)\, dt$$

$$= \int_a^b P(\vec{r}(t))\,\underbrace{x'(t)\,dt}_{dx} + \int_a^b Q(\vec{r}(t))\,\underbrace{y'(t)\,dt}_{dy}$$

$$= \int_C P\,dx + \int_C Q\,dy \quad \text{[New notation.]}$$

$$= \int_C P\,dx + Q\,dy$$

Ex:



$\vec{r}(t) = (\overset{x}{t}, \overset{y}{t^2})$

$0 \le t \le 1$

Evaluate: $\displaystyle\int_C y\,dx + dy$

$$= \int_0^1 t^2 \overbrace{(1\,dt)}^{x'(t)\,dt} + 1\,\overbrace{(2t)\,dt}^{y'(t)\,dt}$$

$$= \int_0^1 t^2 + 2t\,dt = \frac{t^3}{3} + t^2 \Big|_{t=0}^{1} = 4/3.$$

[Same as at end of last time.]

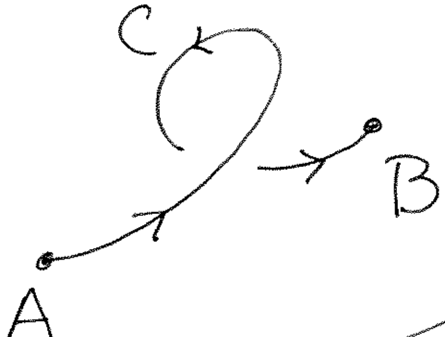
Similarly in $\mathbb{R}^3$: $\int_C P\,dx + Q\,dy + R\,dz$

is just $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$

Fundamental Theorem of Calculus: $f : [a,b] \to \mathbb{R}$ which is differentiable. Then $\int_a^b f'(t)\,dt = f(b) - f(a)$.

Fundamental Theorem of Line Integrals: Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable and C a curve in $\mathbb{R}^n$ from A to B. Then

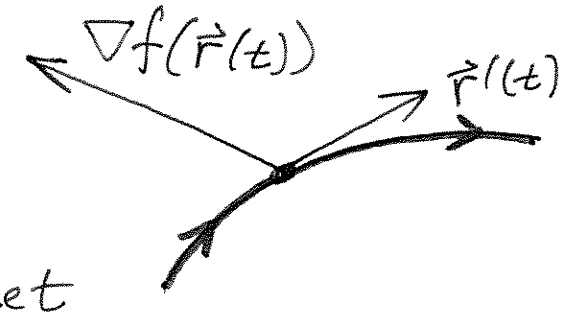
$$\int_C \nabla f \cdot d\vec{r} = f(B) - f(A)$$



Reason: Pick $\vec{r} : [a,b] \to C$ where $\vec{r}(a) = A$ and $\vec{r}(b) = B$.

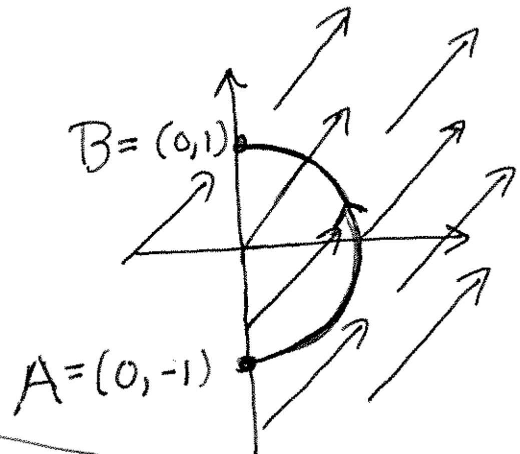
Then
$$\int_C \nabla f \cdot dr = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$$= \int_a^b \underbrace{\left(D_{\vec{r}'(t)} f\right)(\vec{r}(t))}_{\substack{\text{rate } f \text{ changes as we} \\ \text{move along curve at time } t}} \, dt$$



$$= \int_a^b \left(\text{rate of change in } f(t)\right) dt = f(\vec{r}(b)) - f(\vec{r}(a))$$

$$= f(B) - f(A).$$



Ex: $f(x,y) = x + y$ $\quad \vec{F} = \nabla f = (1,1)$

Curve C.

$\vec{r}(t) = (\cos t, \sin t)$

$-\pi/2 \leq t \leq \pi/2$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{-\pi/2}^{\pi/2} (1,1) \cdot \underbrace{(-\sin t, \cos t)}_{\vec{r}'(t)} \, dt$$

$$= \int_{-\pi/2}^{\pi/2} -\sin t + \cos t \, dt$$

$$= \cos t + \sin t \Big|_{t=-\pi/2}^{\pi/2} = 1 - (-1) = 2.$$

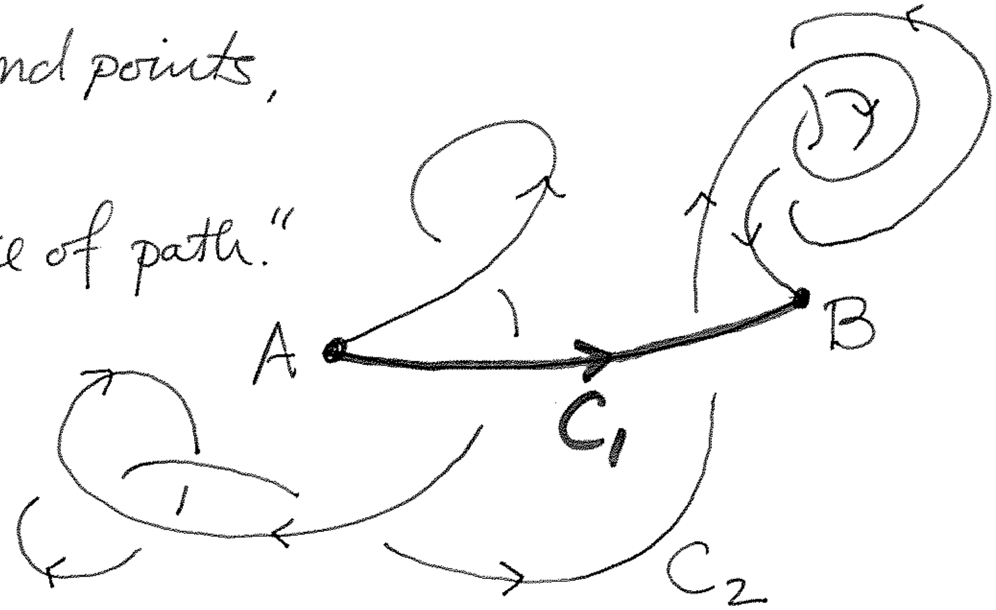
Fund. Thm. says as $\vec{F} = \nabla f$

$$\int_C \vec{F} \cdot d\vec{r} = f(B) - f(A) = 1 - (-1) = 2 \checkmark$$

<u>Consequence</u>: If $\vec{F}$ is conservative ($=\nabla f$)

then $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$ if C_1 and C_2 have the same end points,

"Independence of path."



[Next two lectures will explore conservative vector fields and characterize them by properties of their path integrals.]