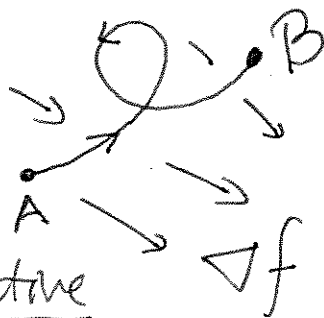


Lecture 20: Conservative Vector Fields (§16.3)

Previously on Math 241:

Fundamental Thm of Line Integrals: $f: \mathbb{R}^n \rightarrow \mathbb{R}$
differentiable and C a curve joining A to B .

Then
$$\int_C \nabla f \cdot d\vec{r} = f(B) - f(A)$$

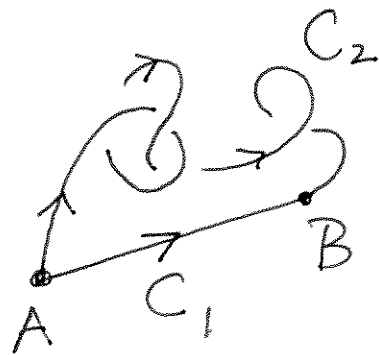


A vector field $\vec{F}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is conservative
when $\vec{F} = \nabla f$ for some $f: \mathbb{R}^n \rightarrow \mathbb{R}$.

By the Fund. Thm, if \vec{F} is conservative then

(a) Independence of path:

If C_1 and C_2 are two paths joining
 A to B , then



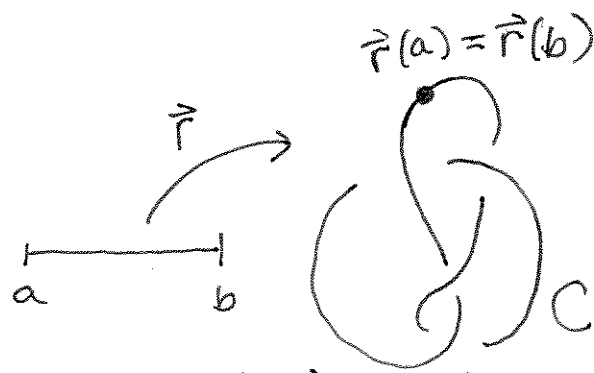
$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$

as both are $f(B) - f(A)$ where $\vec{F} = \nabla f$.

(b) If C is a closed curve (i.e. starts and ends
at the same point, like a circle) then

(2)

$$\int_C \vec{F} \cdot d\vec{r} = 0$$

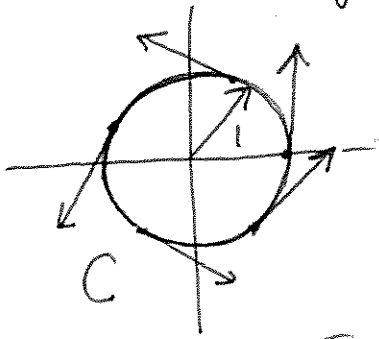


Reason: If $\vec{r}(a) = \vec{r}(b)$ and $\vec{F} = \nabla f$ then

$$\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)) = 0$$

↑ ↑
same input

Ex: $\vec{F} = (-y, x)$ [From 1st lecture on vector fields.]



For the unit circle, $\vec{F} = \vec{T}$ and so

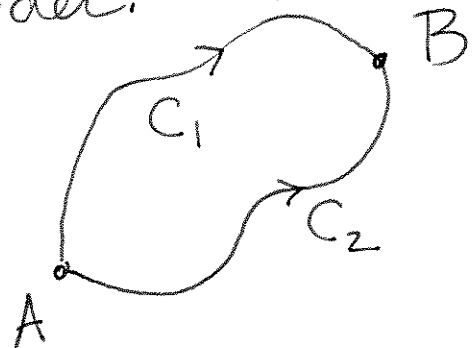
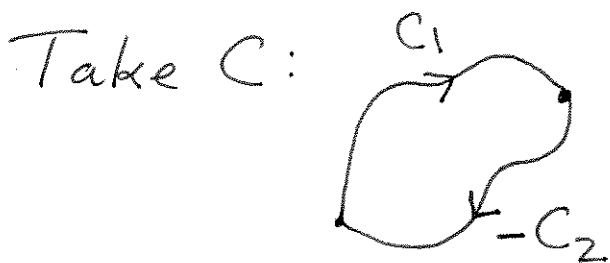
$$\int_C \vec{F} \cdot d\vec{r} = \int_C (\vec{F} \cdot \vec{T}) ds = \int_C ds = 2\pi$$

So by (b), \vec{F} is not conservative.

[On tomorrow's worksheet, will apply (a) in a similar way.]

Note: Conditions (a) and (b) are equivalent. For example, if (b) holds for \vec{F} consider:

Let $-C_2$ be C_2 backwards.



$$\text{Then } 0 = \int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{-C_2} \vec{F} \cdot d\vec{r} \quad (3)$$

$$= \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r} \quad \Rightarrow \quad \int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$

[Conversely, if (a) holds for \vec{F} , then break a closed curve into 2 segments to see that $\int_C \vec{F} \cdot ds = 0$.]

Note: Being conservative is subtle; for example is $\vec{F} = (y, x)$ conservative? [Compare with above ex.]

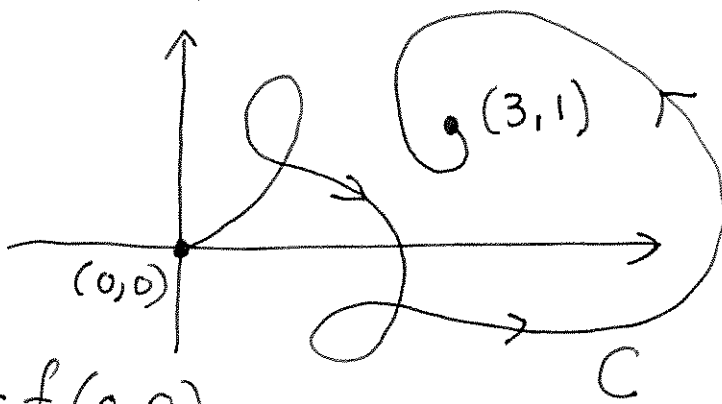
A. Yes. Need $f(x, y)$ with $\frac{\partial f}{\partial x} = y$ and $\frac{\partial f}{\partial y} = x$.

First condition gives $f = \int y dx = xy + C(y)$

Second is similar, so take

$$f = xy.$$

Usefulness is clear:



$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= f(3,1) - f(0,0) \\ &= 3 \cdot 1 - 0 = 3. \end{aligned}$$

[Will give two tests for when \vec{F} is conservative, but first need to introduce some terms...] (4)

Focus on a vector field \vec{F} on \mathbb{R}^2 , with domain D .

Ex: $\vec{F} = (x, y)$ $D = \mathbb{R}^2$

$$\vec{F} = \frac{1}{x^2 + y^2}(-y, x) \quad D = \{(x, y) \neq (0, 0)\}$$

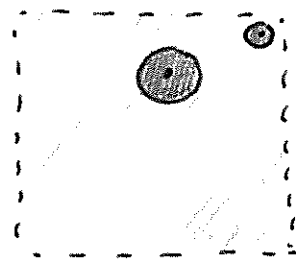
Properties D may have:

Open: Roughly, D contains none of its boundary points. Precisely, each point of D is the center of a disc that is also contained in D .

Ex: $D = \{(x, y) \mid 0 < x < 1, 0 < y < 1\}$

$$D = \mathbb{R}^2$$

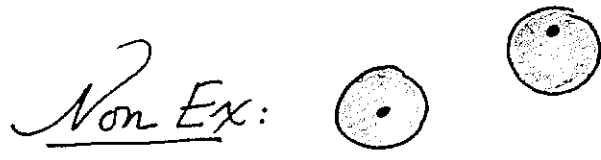
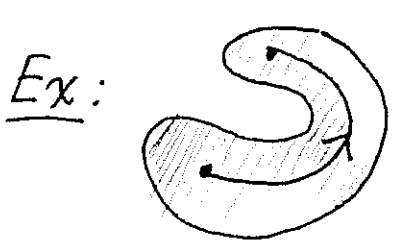
$$D = \{(x, y) \neq \{0, 0\}\}$$



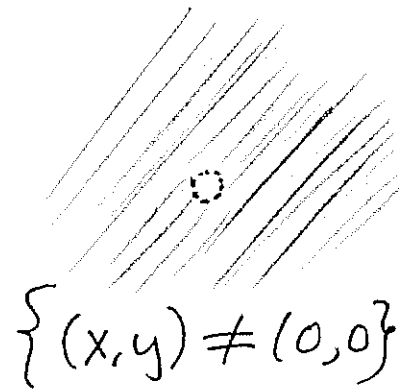
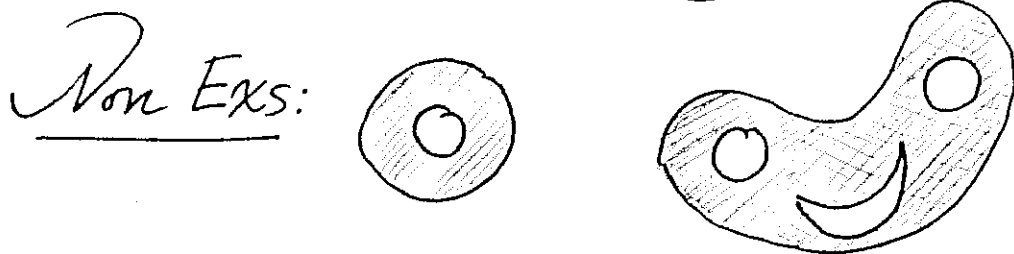
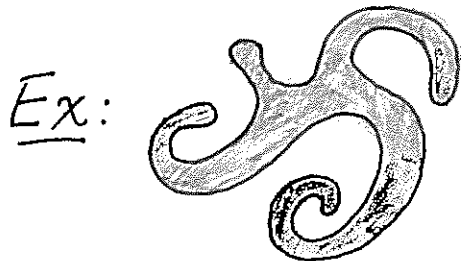
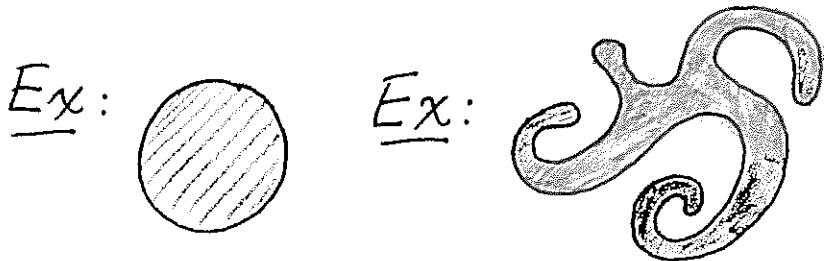
Closed: Opposite of open.

[Precisely, D is closed exactly when its complement is open.]

Connected: Any two points in D can be joined by a path inside D .



Simply Connected: Connected + no holes



Thm A: Suppose \vec{F} is a vector field on an open connected set D in \mathbb{R}^2 . Then \vec{F} is conservative if and only if $\int_C \vec{F} \cdot d\vec{r}$ is path independent.

Thm B: Suppose $\vec{F} = (P, Q)$ is a vector field on an open simply connected set D in \mathbb{R}^2 . Then \vec{F} is conservative if and only if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on D .

[Thm A is nice but hard to apply. Will talk about next time.]

⑥

Reason for Thm B: Suppose $\vec{F} = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$

Then $\frac{\partial P}{\partial y} = \frac{\partial}{\partial y} \frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x}$

and

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x} \frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x \partial y}$$

Point: mixed partials are equal!

[We'll come back to why $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Rightarrow \vec{F}$ conservative in about a month.]

Ex: $\vec{F} = (y, x)$ Then $\frac{\partial P}{\partial y} = 1 = \frac{\partial Q}{\partial x} \Rightarrow$ conservative

$\vec{F} = (-y, x)$ Then $\frac{\partial P}{\partial y} = -1 \neq 1 = \frac{\partial Q}{\partial x} \Rightarrow$ not conser.

Ex: $\vec{F} = \frac{1}{x^2+y^2} (-y, x)$ on $D = \{(x, y) \neq 0\}$

On HW: Check that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ but its not path independent, and hence not conservative.

Point: D is not simply connected.