

# Lecture 21: Conservative vector fields II (§16.3) ①

Last time:  $\vec{F}: \mathbb{R}^n \rightarrow \mathbb{R}^n$  is conservative if  $\vec{F} = \nabla f$  for some  $f: \mathbb{R}^n \rightarrow \mathbb{R}$ .

Thm A: A vector field  $\vec{F}$  on an open connected region  $D$  in  $\mathbb{R}^2$  is conservative if and only if  $\int_C \vec{F} \cdot d\vec{r}$  is path independent.

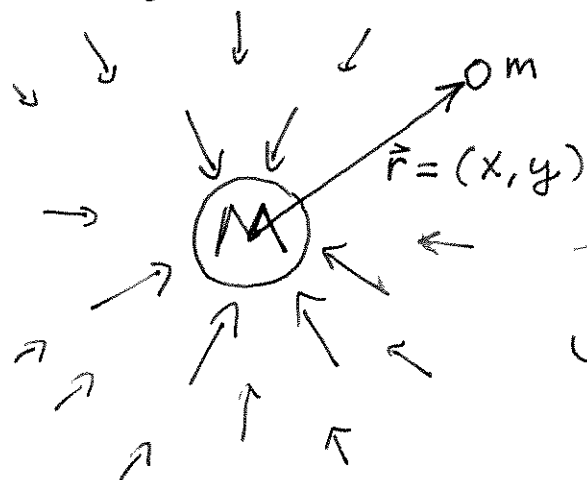
Thm B:  $\vec{F} = (P, Q)$  on a simply connected open  $D$  in  $\mathbb{R}^2$  is conservative if and only if  $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$  on  $D$ .

Thm A Also works for  $\mathbb{R}^n$ .

Thm B Has analogs for  $\mathbb{R}^n$ , but more complicated.

[We will come back to this later, but see §16.3 #29]  
[Note to self: condition is  $H^1(D; \mathbb{R}) = 0$ .]

Physical motivation: Force  $\vec{F} = -\frac{MmG}{|\vec{r}|^3} \vec{r}$



$$V = -\frac{MmG}{|\vec{r}|} \quad \text{Potential function}$$

Note:  $\vec{F} = -\nabla V$

$V$  is larger the farther away  $m$  is and  $\vec{F}$  points in the direction of fastest decrease of  $V$ . (2)

More generally, suppose an object's path is

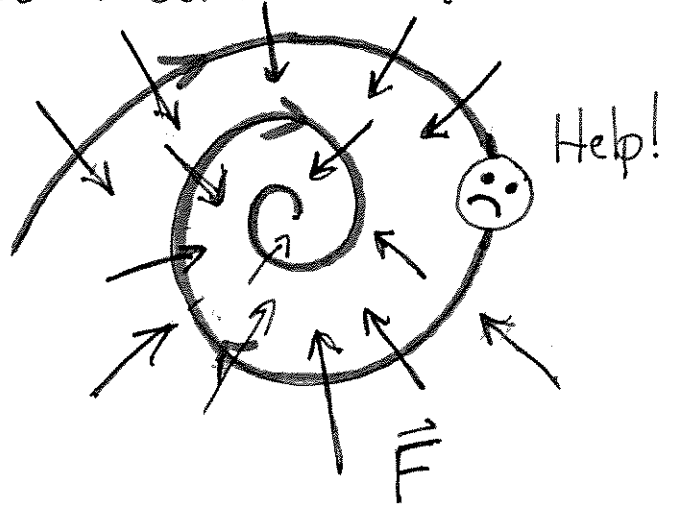
$\vec{r}: \mathbb{R} \rightarrow \mathbb{R}^2$ , acted on by a conservative force

$\vec{F} = -\nabla V$  for  $V: \mathbb{R}^2 \rightarrow \mathbb{R}$ .

Newton's Law:  $F = ma$

or

$$\vec{F}(\vec{r}(t)) = m \vec{r}''(t)$$



Total Energy:

$$E(t) = \left( \begin{array}{c} \text{Kinetic} \\ \text{energy} \end{array} \right) + \left( \begin{array}{c} \text{Potential} \\ \text{energy} \end{array} \right)$$

$$= \frac{1}{2} m |\dot{\vec{r}}(t)|^2 + V(\vec{r}(t))$$

Conservation of Energy:  $E(t)$  is constant, independent of  $t$ .

Reason: Let's compute  $E'(t)$ . If  $\vec{r} = (r_1, r_2)$ ,

Then

$$\begin{aligned} \frac{d}{dt} \left( \frac{1}{2} m |\vec{r}'(t)|^2 \right) &= \frac{1}{2} m \frac{d}{dt} \left( (r_1'(t))^2 + (r_2'(t))^2 \right) \quad (3) \\ &= \frac{1}{2} m \left( 2r_1'(t)r_1''(t) + 2r_2'(t)r_2''(t) \right) \\ &= m \vec{r}''(t) \cdot \vec{r}'(t) \end{aligned}$$

Also

$$\begin{aligned} \frac{d}{dt} V(\vec{r}(t)) &= \frac{\partial V}{\partial x}(\vec{r}(t)) r_1'(t) + \frac{\partial V}{\partial y}(\vec{r}(t)) r_2'(t) \\ &= \nabla V(\vec{r}(t)) \cdot \vec{r}'(t) \end{aligned}$$

So

$$\begin{aligned} E'(t) &= m \vec{r}''(t) \cdot \vec{r}'(t) + \nabla V(\vec{r}(t)) \cdot \vec{r}'(t) \\ &= \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) - \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \\ &= 0 \end{aligned}$$

Hence energy is conserved!

Thm A: A vector field  $\vec{F}$  on an open connected region  $D$  in  $\mathbb{R}^2$  is conservative if and only if  $\int_C \vec{F} \cdot d\vec{r}$  is path independent.

Reason for Thm A: Suppose  $\int_C \vec{F} \cdot d\vec{r}$  is path independent. Pick a point  $A_0$  in  $D$ .

Define  $f: D \rightarrow \mathbb{R}$  by

$$f(B) = \int_C \vec{F} \cdot d\vec{r}$$

where  $C$  is any path from  $A_0$  to  $B$  inside of  $D$ .

[This is an odd sort of function, but it is a way of associating a number to each point...]

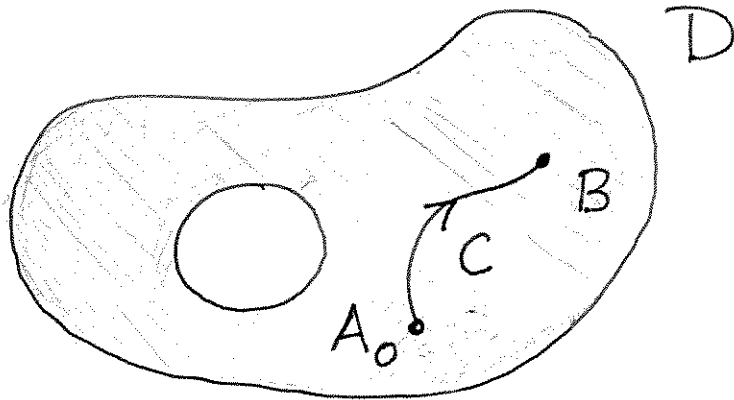
Note:  $f(A_0) = 0$  and so, the F.T.L.I. says

$$\int_C \nabla f \cdot d\vec{r} = f(B) - f(A_0) = f(B)$$

Point:  $\nabla f = \vec{F}$ . For instance, let's compute

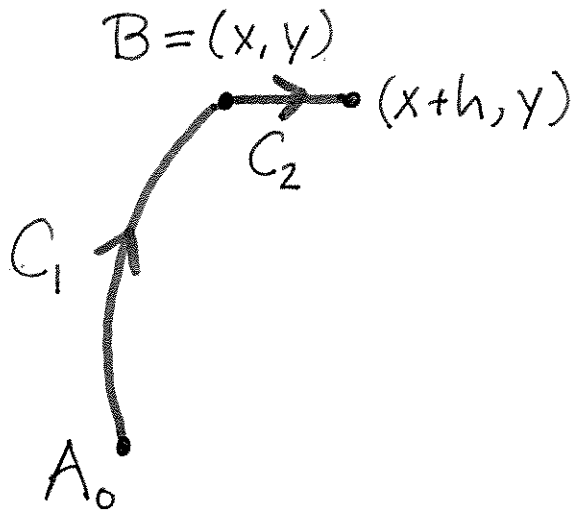
$$\frac{\partial f}{\partial x}(x, y) = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$$

Now consider how we compute these:



(4)

(5)



$$f(x, y) = \int_{C_1} \vec{F} \cdot d\vec{r}$$

$$f(x+h, y) = \int_{C_1 + C_2} \vec{F} \cdot d\vec{r}$$

Thus

$$\frac{\partial f}{\partial x}(x, y) = \lim_{h \rightarrow 0} \frac{1}{h} \int_{C_2} \vec{F} \cdot d\vec{r}$$

Now if  $\vec{F} = (P, Q)$  then as  $\vec{T} = (1, 0)$  we have

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot \vec{T} \, ds = \int_{C_2} P \, ds$$

So

$$\frac{\partial f}{\partial x}(x, y) = \lim_{h \rightarrow 0} \underbrace{\frac{1}{h} \int_{C_2} P \, ds}_{\text{Average of } P \text{ on } C_2} = P(x, y)$$

As averaging over shorter and shorter curves.

Likewise,  $\frac{\partial f}{\partial y} = Q$  and so  $\nabla f = \vec{F}$ , that is  $\vec{F}$  is conservative.