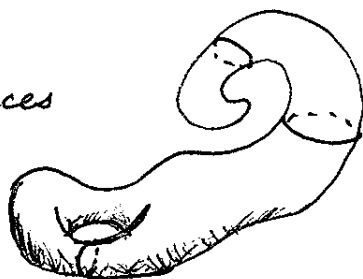


# Lecture 22: Multivariable Integration (§15.1)

Curves: 

$$\int_C f ds \quad \int_C \vec{F} \cdot d\vec{r}$$

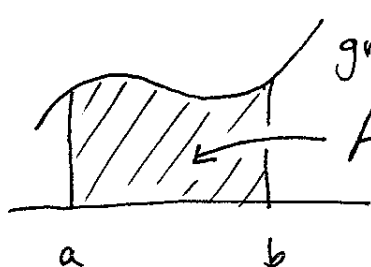
Goal: Surfaces



$$\int_S f dA$$

[Things like parametrization, integration will repeat for surfaces...]

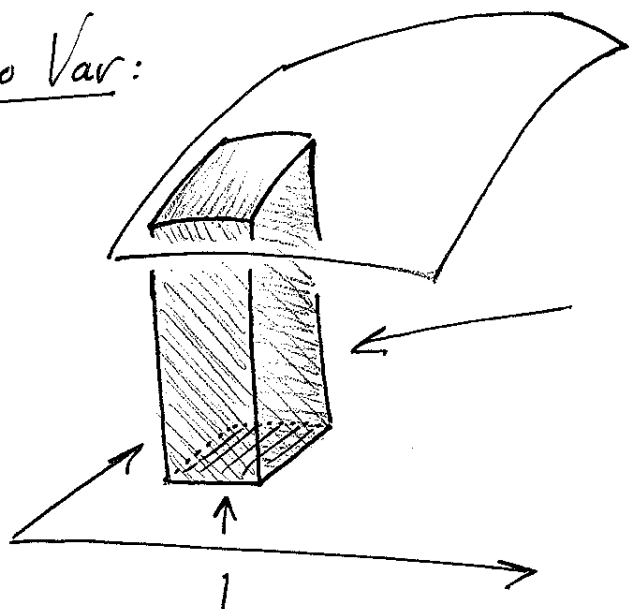
One Var:



$$\text{Area} = \int_a^b f(x) dx$$

[Computed using the Fund. Thm. of Calc.]

Two Var:



graph of  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$

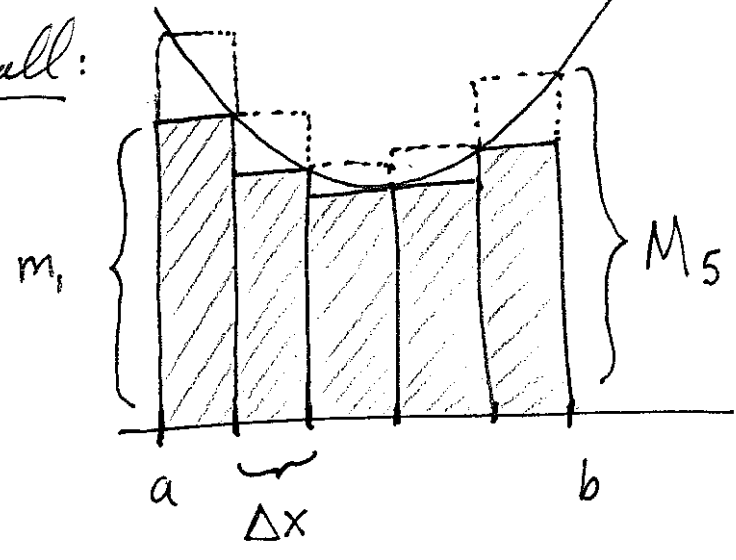
$$\text{Volume is } \iint_R f(x,y) dA$$

Base  $R$ , a square

Q1: How do we formulate volume mathematically?

Q2: How do we compute it?

Recall:



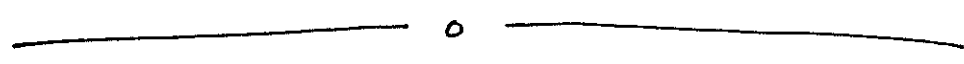
$m_i = \text{min of } f \text{ on } i^{\text{th}} \text{ subinterval}$

$M_i = \text{max of } f \text{ on } i^{\text{th}} \text{ subinterval.}$

Then

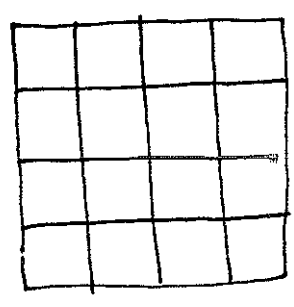
$$\sum_{i=1}^n m_i \Delta x \leq \int_a^b f(x) dx \leq \sum_{i=1}^n M_i \Delta x$$


If  $f$  is continuous, then as  $\Delta x \rightarrow 0$  the two bounds converge to the thing in the middle.

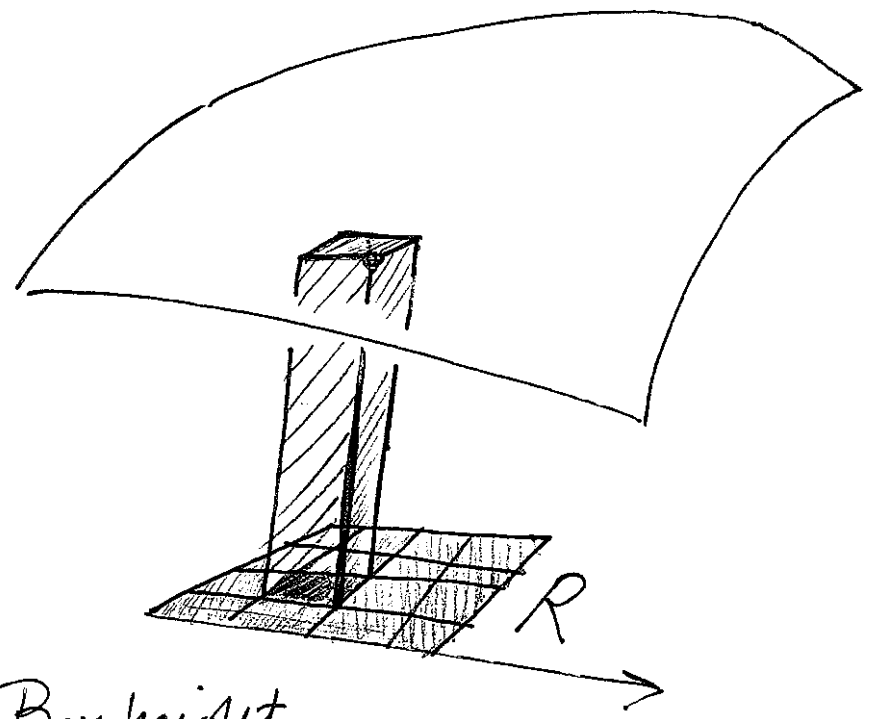


Two Var

R:



Each square  has area  $\Delta x \Delta y$ .



Box height = min of  $f$  on the subsquare.

Thus

$$\sum_{\text{small squares}} \left( \begin{array}{l} \text{min of } f \\ \text{on subsquare} \end{array} \right) \Delta x \Delta y \leq \iint_R f(x,y) dA$$

$$\leq \sum_{\text{small squares}} \left( \begin{array}{l} \text{max of } f \\ \text{on subsquare} \end{array} \right) \Delta x \Delta y$$

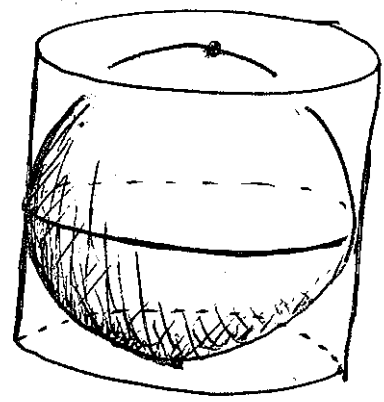
As  $\Delta x, \Delta y \rightarrow 0$  these bounds converge to define the integral. [Provided  $f$  is continuous.]

Q2: How do we compute it?

Archimedes (225 B.C.E.)

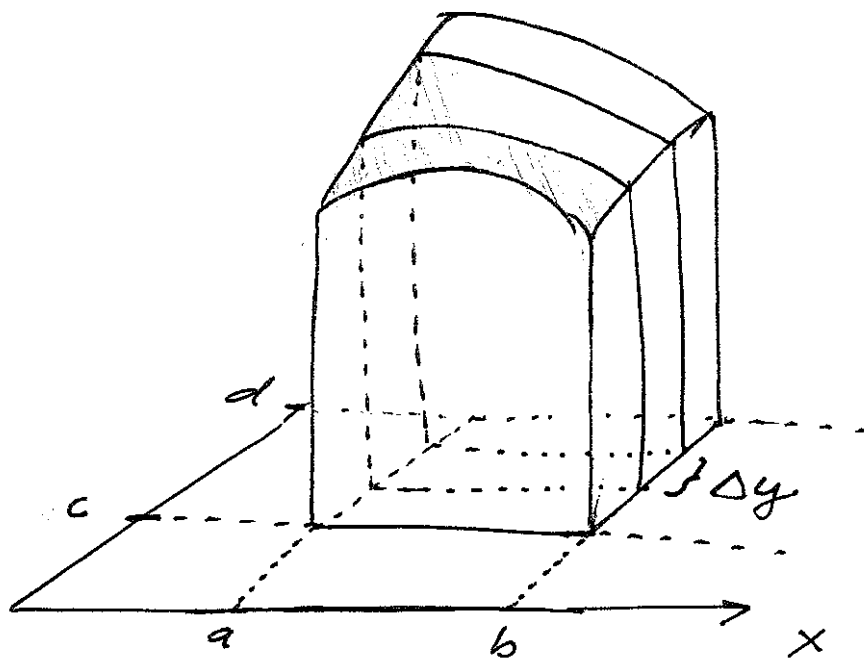
3 : 2

volume and surface area!

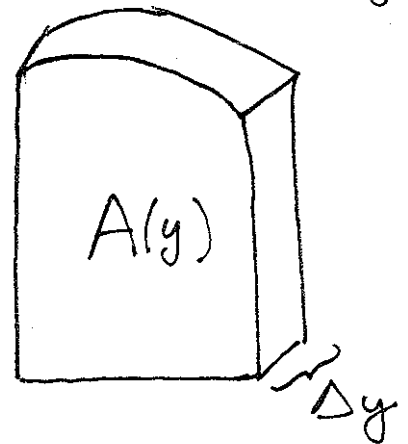


Key: Reduce to one var integrals by slicing.

Let  $A(y)$  be the area of the cross section with the given  $y$  coord.



Volume of a slice  
is  $\approx A(y)\Delta y$



Add to get  
total volume

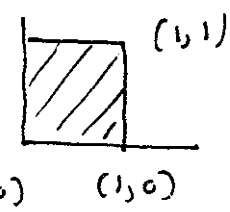
So

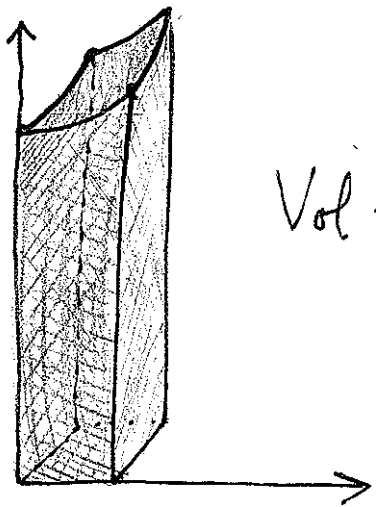
$$\iint_R f(x,y) dA = \int_c^d A(y) dy$$

$$= \int_c^d \left( \int_a^b f(x,y) dx \right) dy$$

y is fixed

$$= \int_a^b \left( \int_c^d f(x,y) dy \right) dx.$$

Ex:  $f(x,y) = x^2 + y^2 + 5$       $R =$  

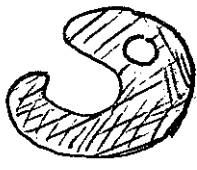


$$\text{Vol} = \iint f(x,y) dA = \int_0^1 \left( \int_0^1 (x^2 + y^2 + 5) dx \right) dy$$

$$= \int_0^1 \left( \frac{x^3}{3} + (y^2 + 5)x \Big|_{x=0}^{x=1} \right) dy$$

$$= \int_0^1 \left( \frac{1}{3} + (y^2 + 5) \right) dy = \frac{16}{3}y + \frac{y^3}{3} \Big|_{y=0}^{y=1} = \frac{17}{3} = 5 \frac{2}{3}$$

Next time: Dealing with regions  $R$  which are not rectangles

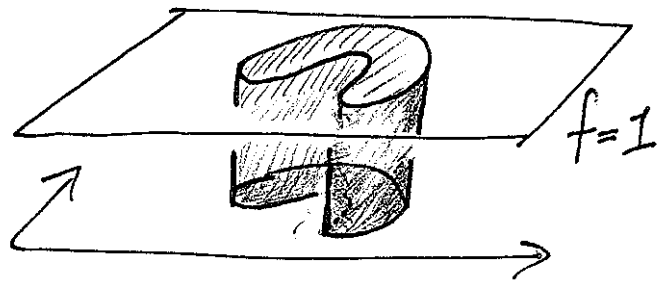


Idea: Break into small boxes.

Other meanings of the integral:

①  $\iint_R 1 dA = \text{Area of } R$

$dA = d(\text{Area})$



$\text{Vol} = (\text{height}) \cdot (\text{Area})$

② Average of  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$  on  $R$  is

$$\text{Average} = \frac{1}{\text{Area}(R)} \iint_R f \, dA$$

③  $R$  made of material with density given by  $\rho: \mathbb{R}^2 \rightarrow \mathbb{R}$ . Then

$$\text{Total Mass} = \iint_R \rho \, dA.$$