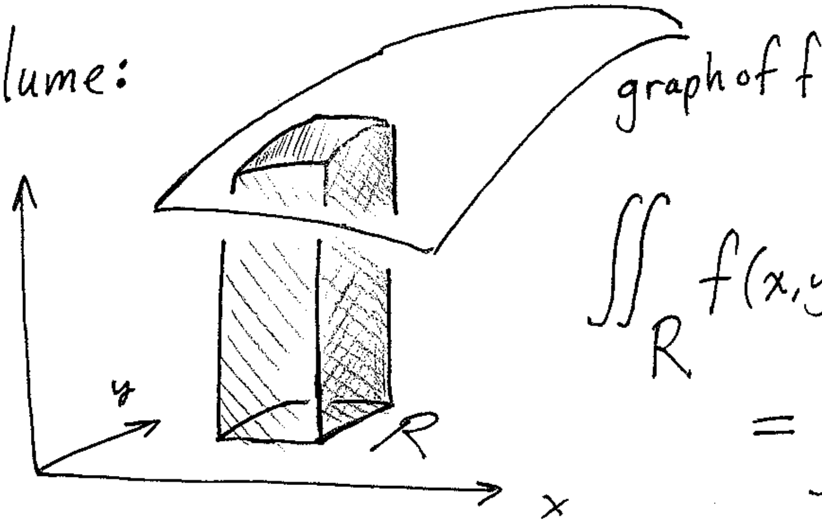
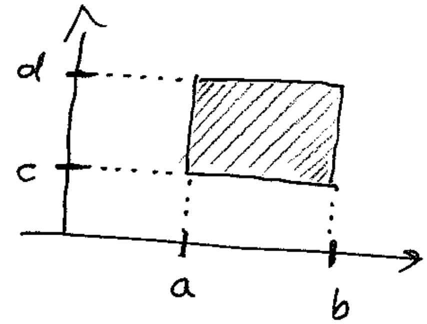
Lecture 23: Integrating over more complicated regions (§15.2–3)

Last time: $f: \mathbb{R}^2 \to \mathbb{R}$ R = rectange in $\mathbb{R}^2$.

Volume:

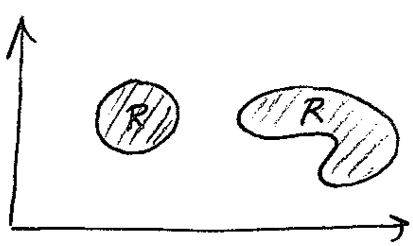


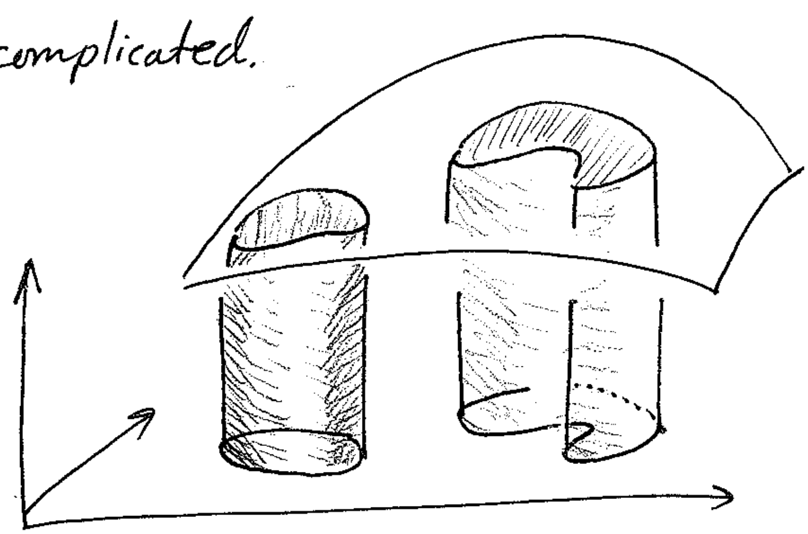


$$\iint_R f(x,y)\, dA = \int_c^d \left(\int_a^b f(x,y)\, dx \right) dy$$

[Other interpretions: averages, total mass, center of mass, ...]

Today: When R is more complicated.

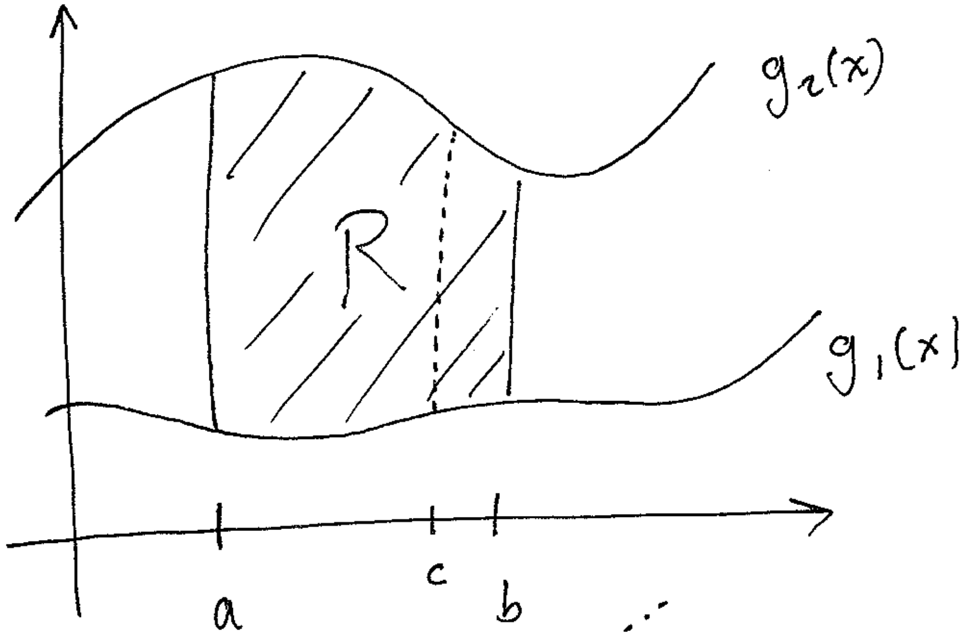




Q: How to compute

$$\text{Volume} = \iint_R f(x,y)\, dA \,?$$

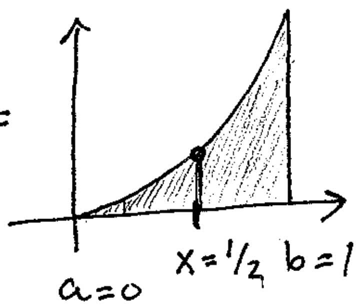
Suppose R has the form:



$$R = \left\{ \begin{array}{l} a \le x \le b \ \underline{\text{and}} \\ g_1(x) \le y \le g_2(x) \end{array} \right\}$$

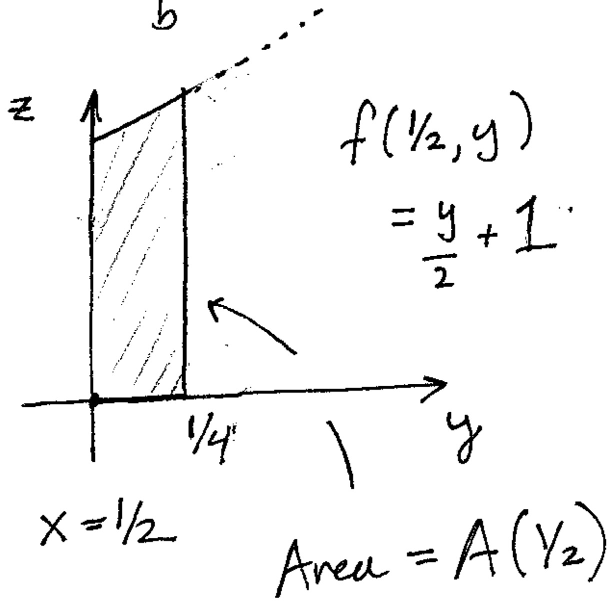
Idea: Slice along lines $x = c$.

Ex: $R =$



$g_2 = x^2$

$g_1 = 0$



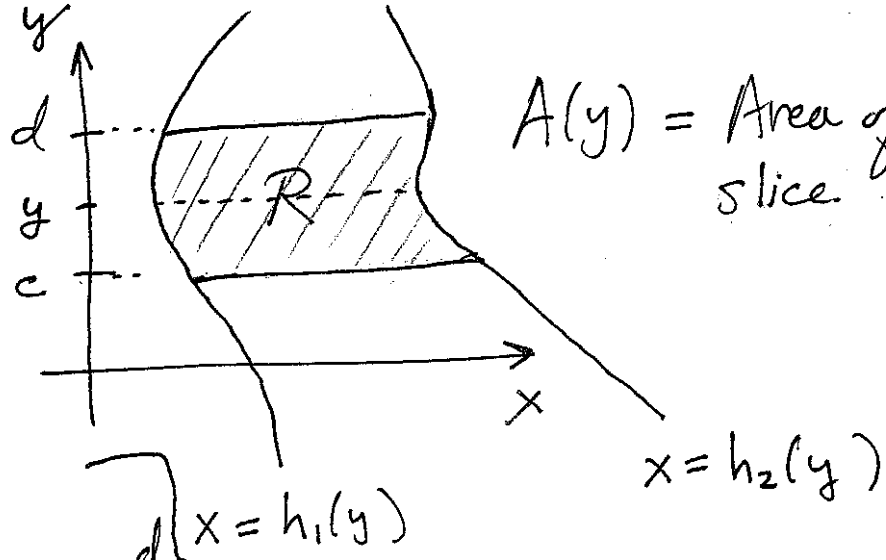
$$\iint_R \underbrace{xy + 1}_{f(x,y)} \, dA = \int_0^1 A(x)\, dx$$

$$= \int_0^1 \left(\int_0^{x^2} xy + 1 \, dy \right) dx = \int_0^1 \left(\frac{xy^2}{2} + y \Big|_{y=0}^{y=x^2} \right) dx$$

$$= \int_0^1 \frac{1}{2} x (x^2)^2 + x^2 \, dx = \int_0^1 \frac{1}{2} x^5 + x^2 \, dx$$

$$= \frac{1}{12} x^6 + \frac{x^3}{3} \Big|_{x=0}^{1} = \frac{1}{12} + \frac{1}{3} = \frac{5}{12}$$

Similar case:

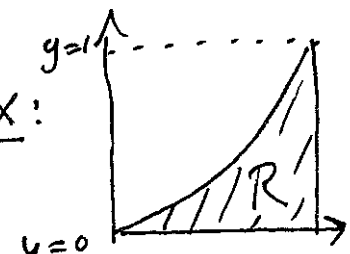


$A(y) =$ Area of slice

$$\iint_R f(x,y)\, dA = \int_c^d A(y)\, dy$$

$$= \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\, dx \right) dy$$

Ex:



$$h_1(y) = \sqrt{y}$$
$$h_2(y) = 1$$

$$\iint_R xy + 1\, dA = \int_0^1 A(y)\, dy = \int_0^1 \left(\int_{\sqrt{y}}^{1} xy + 1\, dx \right) dy$$

$$= \int_0^1 \left(\frac{1}{2}x^2 y + x \Big|_{x=\sqrt{y}}^{x=1} \right) dy = \int_0^1 \left(\frac{1}{2}y + 1\right) - \left(\frac{1}{2}y^2 + \sqrt{y}\right) \cdot dy$$

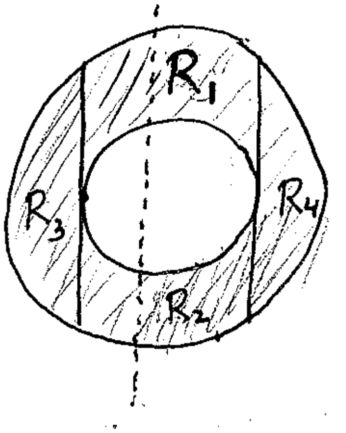
$$= \int_0^1 -\frac{1}{2}y^2 + \frac{1}{2}y - \sqrt{y} + 1\, dy = -\frac{y^3}{6} + \frac{y^2}{4} - \frac{2}{3}y^{3/2} + y \Big|_{y=0}^{1}$$

$$= -\frac{1}{6} + \frac{1}{4} - \frac{2}{3} + 1 = \frac{-2 + 3 - 8 + 12}{12} = \frac{5}{12}$$ ✓

[Book calls these two kinds of regions type I and type II. Some regions are both, in which case it can be easier to do things one way or the other.]

General region:

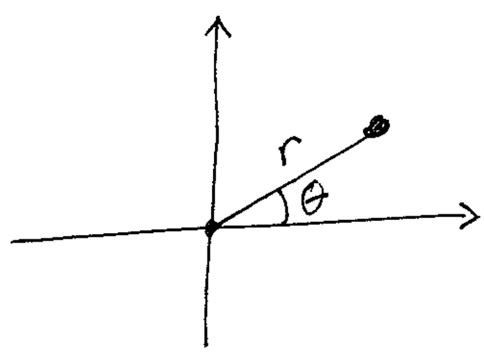
(A) Cut into simple pieces.



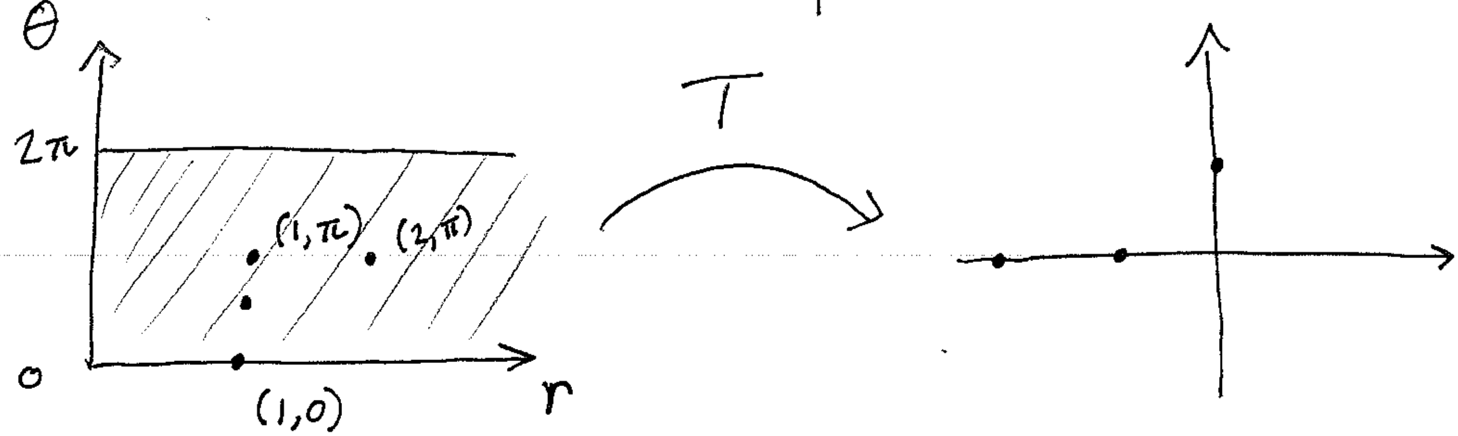
$$\iint_R f\,dA = \sum_{i=1}^{4} \iint_{R_i} f\,dA$$

(B) Change Coordinates, so R becomes easier to describe.

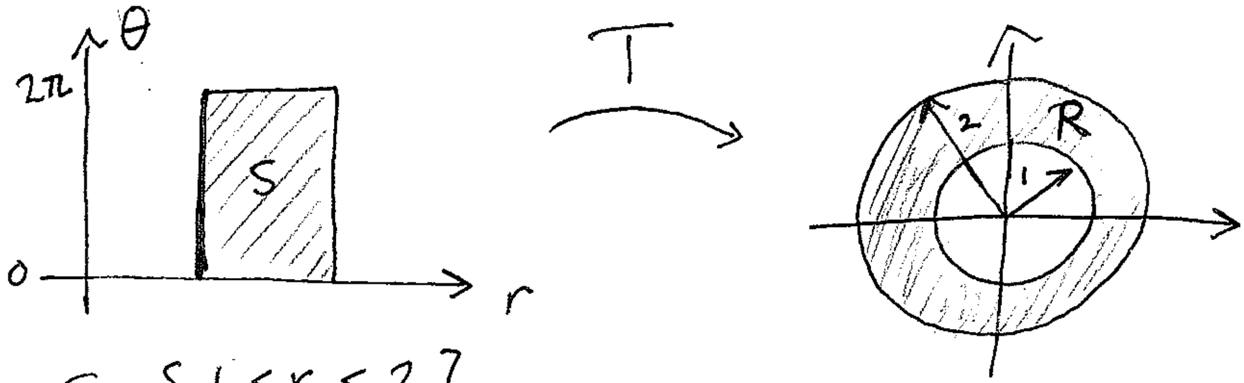
Polar Coordinates:



$x = r\cos\theta$
$y = r\sin\theta$



$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad T(r,\theta) = (r\cos\theta,\ r\sin\theta)$$



$$S = \left\{ \begin{matrix} 1 \leq r \leq 2 \\ 0 \leq \theta \leq 2\pi \end{matrix} \right\}$$

$$\boxed{R = T(S).}$$

<u>Goal</u>: Relate $\iint_R f\, dA$ to an integral over S.

<u>First try</u>:

Guess!

$$\iint_R 1\, dA \stackrel{?}{=} \iint_S 1\, dA = \int_0^{2\pi}\int_1^2 1\, dr\, d\theta$$

$$= \int_0^{2\pi} r\Big|_{r=1}^{2}\, d\theta = \int_0^{2\pi} 1\, d\theta = 2\pi.$$

<u>Is this right?</u>

$$\iint_R 1\, dA = \text{Area}(R) = \text{Area}\begin{pmatrix}\text{disc of}\\ \text{rad } 2\end{pmatrix} - \text{Area}\begin{pmatrix}\text{disc of}\\ \text{rad } 1\end{pmatrix}$$

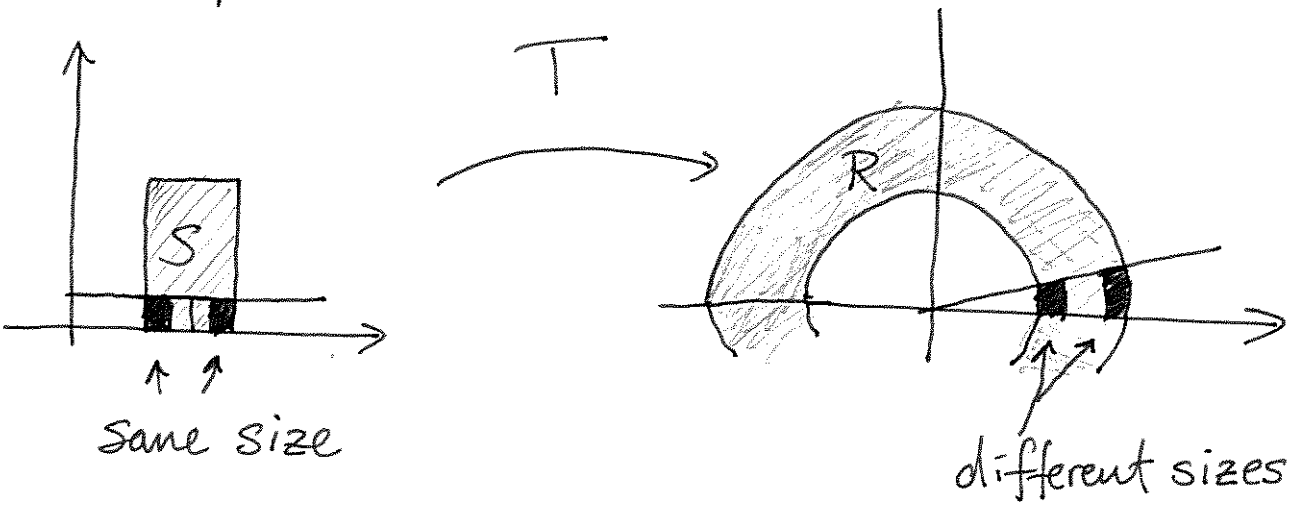
$$= \pi 2^2 - \pi 1^2 = 3\pi.$$

So this didn't work...

Source of problem: T distorts area

(in a nonuniform way).

For example:



Solution: In polar coordinates $dA = r\,dr\,d\theta$

[Will explain in detail next lecture, but for now let's just see that it works in this case.]

$$\iint_S 1\,dA = \int_0^{2\pi}\int_1^2 1\cdot r\,dr\,d\theta = \int_0^{2\pi}\left(\frac{1}{2}r^2\Big|_{r=1}^{r=2}\right)d\theta$$

$$= \int_0^{2\pi}\frac{1}{2}(2^2-1^2)\,d\theta = \frac{3}{2}\int_0^{2\pi}1\,d\theta$$

$$= \frac{3}{2}\cdot 2\pi = 3\pi$$

which is indeed the same as $\iint_R 1\,dA$!