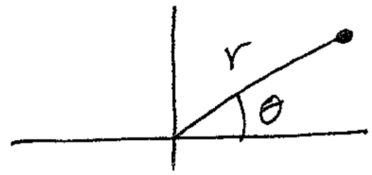
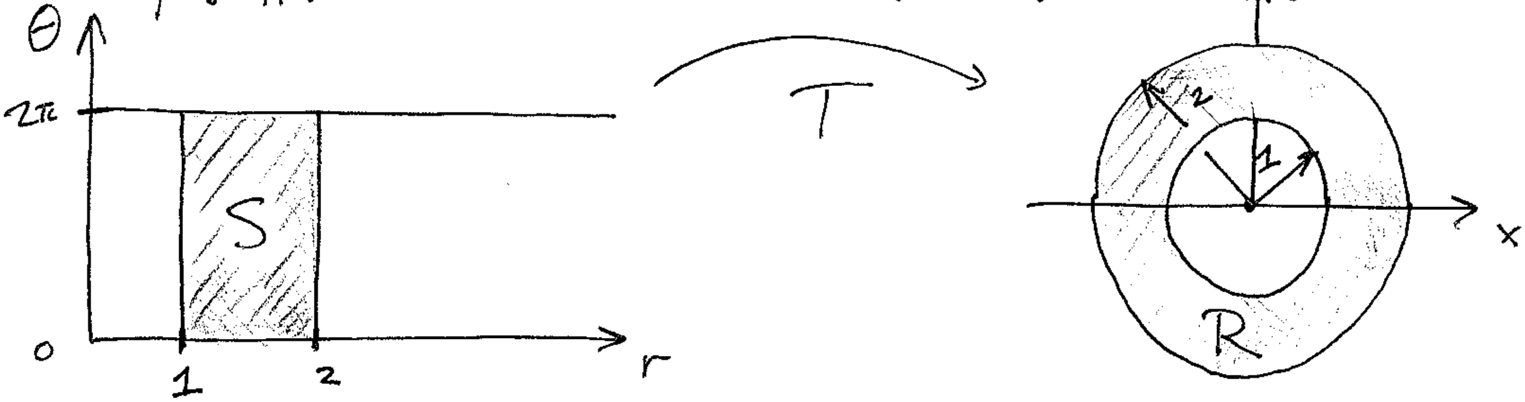
Lecture 23: Polar coordinates (§15.3) and applications (§15.4)

$x = r\cos\theta$
$y = r\sin\theta$

<u>Last time:</u> Polar coordinates



$$T : \mathbb{R}^2 \to \mathbb{R}^2 \quad T(r,\theta) = (r\cos\theta, r\sin\theta)$$



$R = T(S)$

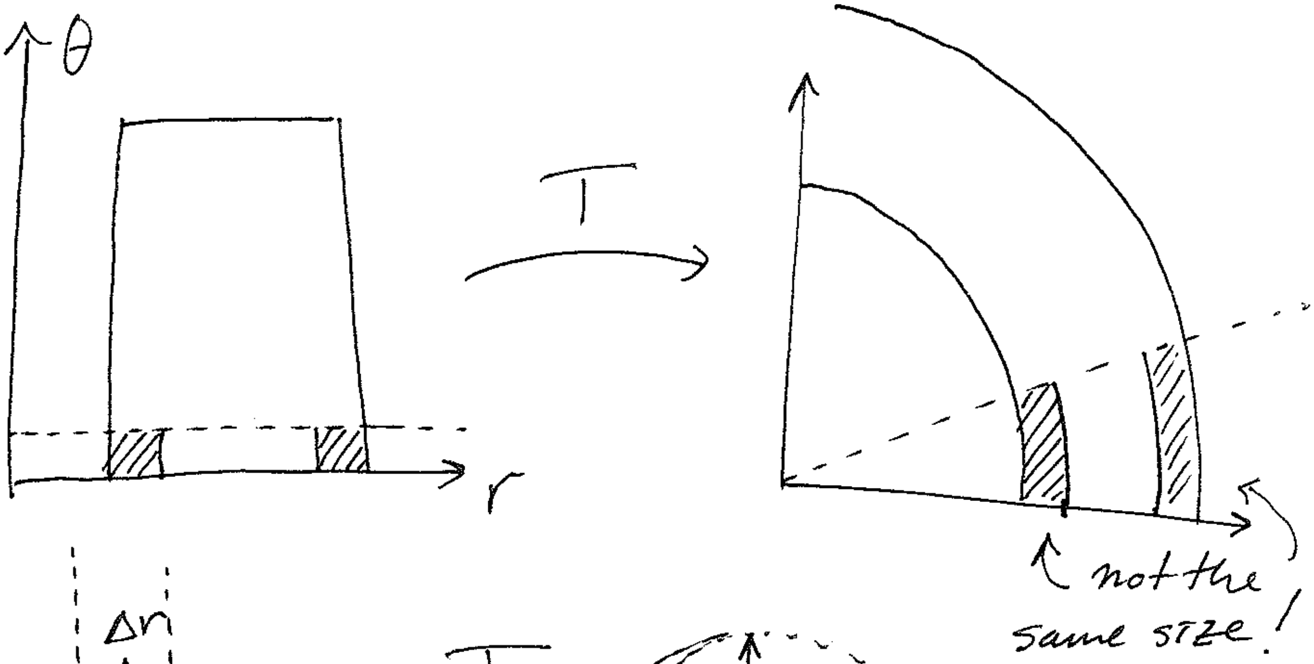
<u>Goal:</u> Relate $\iint_R f\,dA$ to an integral over S.

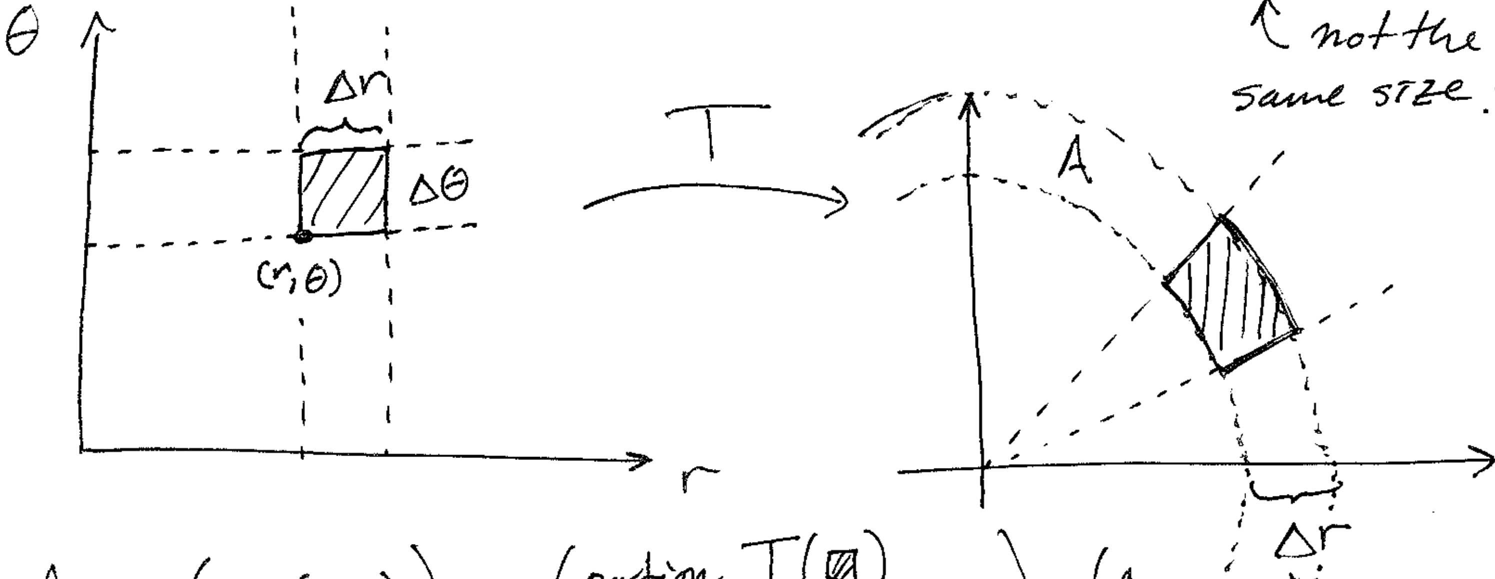
Guess!

<u>First try:</u>
$$\iint_R 1\,dA \stackrel{?}{=} \iint_S 1\,dA = \int_0^{2\pi}\int_1^2 1\,dr\,d\theta$$
$$= \int_0^{2\pi} r\Big|_{r=1}^{2} d\theta = \int_0^{2\pi} 1\,d\theta = 2\pi.$$

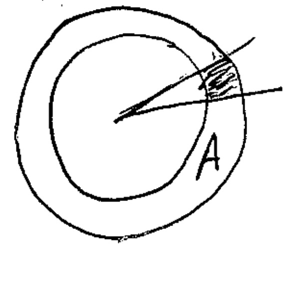
<u>Is this right?</u>
$$\iint_R 1\,dA = \text{Area}(R) = \text{Area}\begin{pmatrix}\text{disc of}\\ \text{rad } 2\end{pmatrix} - \text{Area}\begin{pmatrix}\text{disc of}\\ \text{rad } 1\end{pmatrix}$$
$$= \pi\cdot 2^2 - \pi 1^2 = 3\pi \qquad \text{So } \underline{\text{NO}}.$$

Issue:





$$\text{Area}(T(\blacksquare)) = \left(\begin{array}{c}\text{portion } T(\blacksquare) \\ \text{is of the annulus } A\end{array}\right) \cdot \left(\begin{array}{c}\text{Area of} \\ A\end{array}\right)$$



$$= \frac{\Delta\theta}{2\pi}\Big(\underbrace{\pi(r+\Delta r)^2 - \pi r^2}_{\pi(r^2 + 2r\Delta r + \Delta r^2 - r^2)}\Big)$$

$$= r\,\Delta r\,\Delta\theta + \tfrac{1}{2}\Delta r^2\Delta\theta \approx r\,\Delta r\,\Delta\theta \text{ if } \Delta r \text{ is small.}$$

$$\text{Area}(R) = \sum_{\substack{\text{subrects} \\ \text{of } S}} \text{Area}(T(\blacksquare)) \approx \sum r\,\Delta r\,\Delta\theta$$

$$\approx \iint_S r\, dA$$

So

$$\text{Area}(R) = \iint_R 1\, dA = \iint_S r\, dA = \int_0^{2\pi}\int_0^{1} r\, dr\, d\theta$$

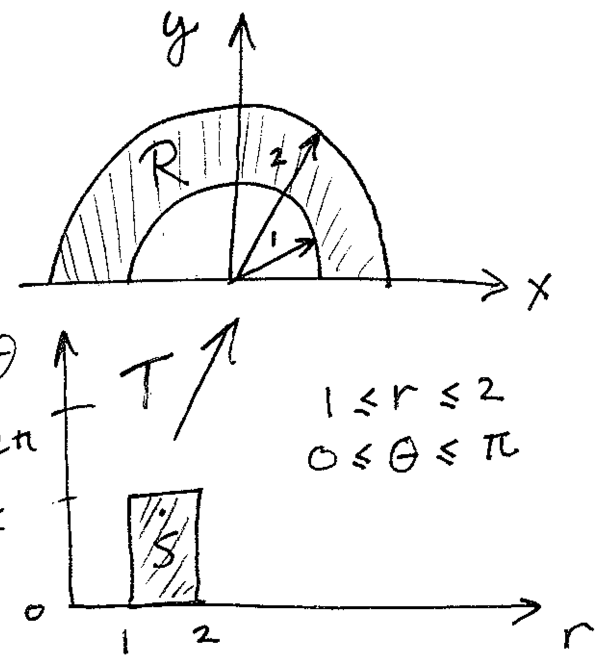
$$= \int_0^{2\pi} \frac{r^2}{2}\Big|_{r=1}^{2} d\theta = \int_0^{2\pi} \frac{2^2}{2} - \frac{1^2}{1}\, d\theta$$

$$= \int_0^{2\pi} \frac{3}{2}\, d\theta = 3\pi$$

which matches our geometric answer!

Summary: In polar coordinates, $dA = r\,dr\,d\theta$

not $dr\,d\theta$.



Ex: $\iint_R f(x,y)\, dA$ for $f(x,y) = y$.

$$= \iint_S f(T(r,\theta))\, r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_1^{2} (r\sin\theta)\, r\, dr\, d\theta$$

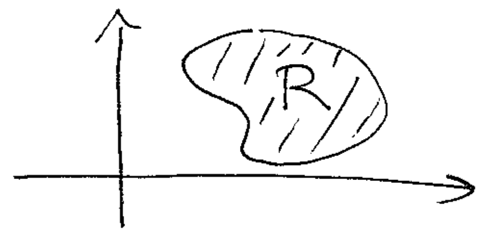
$$= \int_0^{\pi} \frac{r^3}{3}\sin\theta \Big|_{r=1}^{r=2} d\theta$$

$$\int_0^{\pi} \frac{1}{3}\sin\theta\,(2^3-1^3)\,d\theta = \frac{7}{3}\int_0^{\pi} \sin\theta\,d\theta$$

$$= \frac{7}{3}(-\cos\theta)\Big|_{\theta=0}^{\pi} = \frac{14}{3}.$$

Challenge: Get the same answer w/o polar coor!

Applications: (Section 15.5).



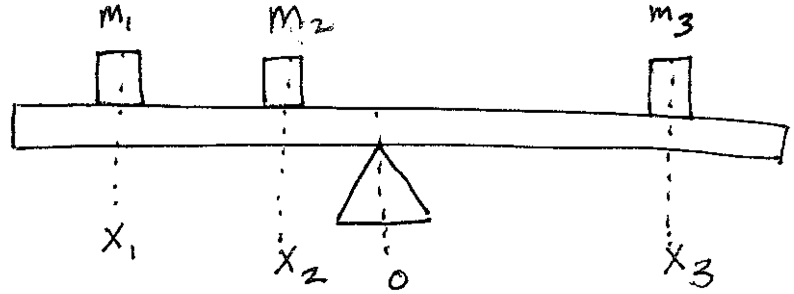
$$\text{Area} = \iint_R 1\,dA$$

$$\text{Mass} = \iint_R \rho(x,y)\,dA \quad \text{where } \rho(x,y) = \frac{\text{mass}}{\text{area}}$$

$$\text{Average} = \frac{1}{\text{Area}(R)}\iint f\,dA$$

[Already mentioned these, are just like line int case. As are the next ones, but we never actually talked about these in detail. Let's do that now...]

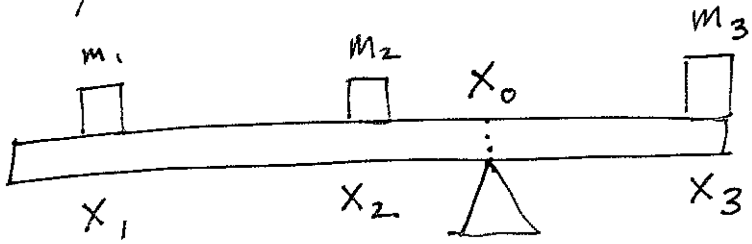
Center of mass:



Balanced means:

$$m_1 x_1 + m_2 x_2 + m_3 x_3 = 0$$

Suppose we want to find the balance point, i.e. solve for x_0



where

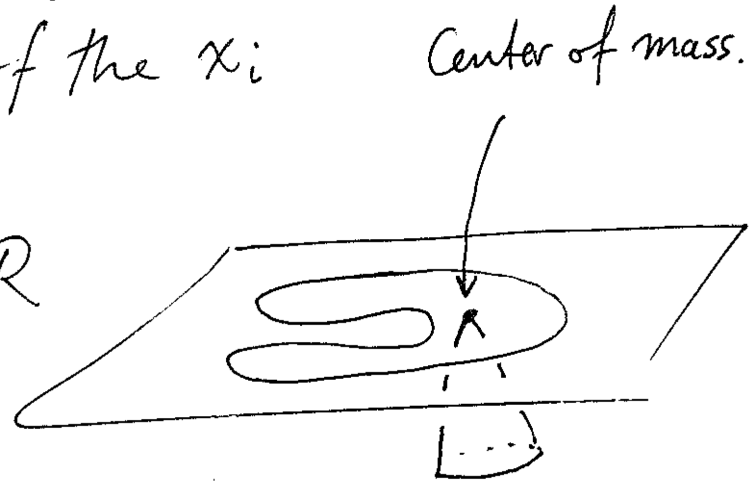
$$m_1(x_1 - x_0) + m_2(x_2 - x_0) + m_3(x_3 - x_0) = 0$$

$$\Rightarrow m_1 x_1 + m_2 x_2 + m_3 x_3 = (m_1 + m_2 + m_3)\, x_0$$

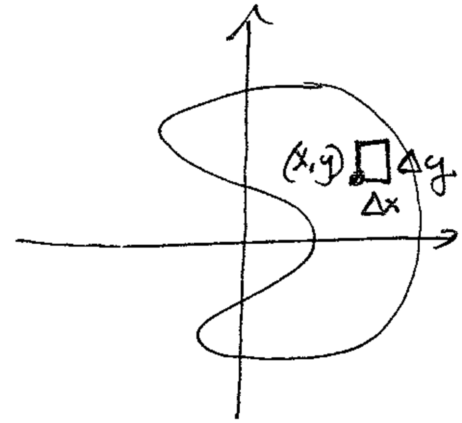
$$\Rightarrow x_0 = \frac{1}{\text{total mass}} \left(m_1 x + m_2 x_2 + m_3 x_3 \right)$$ "Center of mass"

Note: If the m_i are all equal, then x_0 is just the average of the x_i

Now consider a region R in $\mathbb{R}^2$ with density given by $\rho(x,y)$



$$M_x = \frac{1}{\text{Mass}} \sum_{\substack{\text{rect making} \\ \text{up } R}} x \begin{pmatrix}\text{mass of} \\ \text{rect}\end{pmatrix}$$



$$= \frac{1}{\text{Mass}} \sum_{\text{boxes}} x \left(\rho(x,y) \Delta x \Delta y \right)$$

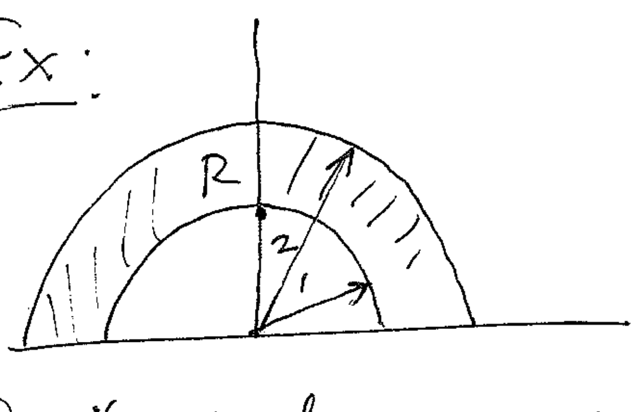
$$\approx \frac{1}{\text{Mass}} \iint_R x \, \rho(x,y) \, dA = M_x$$

<u>Note</u>: If ρ is constant $= \rho_0$, then

$$M_x = \frac{1}{\iint_R \rho_0 \, dA} \iint_R x \rho_0 \, dA = \frac{1}{\rho_0} \frac{1}{\iint_R dA} \rho_0 \iint_R x \, dA$$

$$= \frac{1}{\text{Area}(R)} \iint_R x \, dA = \text{Average of } x \text{ coor on } R.$$

<u>Ex</u>:



Density = const.

$m_x = 0$ by symmetry.

$$m_y = \underset{\text{of } y}{\text{Average}} = \frac{1}{\text{Area}} \iint_R y \, dA$$

$$= \frac{1}{3\pi/2} \cdot \left(\frac{14}{3}\right) = \frac{28}{9\pi}$$

CANT QUITE BALANCE IT!

$$\approx 0.9902$$