

Last time: Cylindrical coordinates

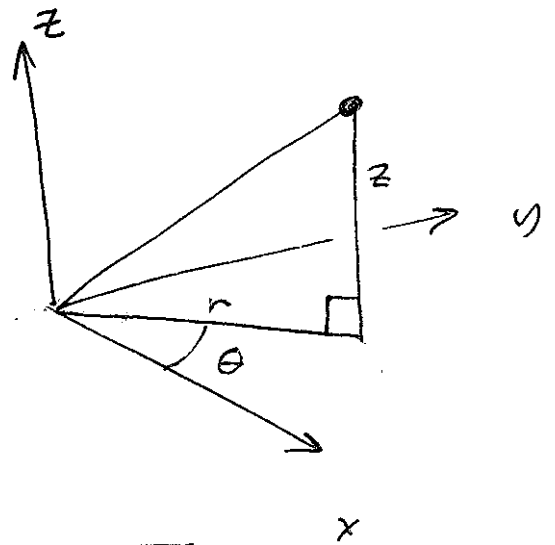
$$(r, \theta, z) \text{ with } 0 \leq r$$

$$0 \leq \theta \leq 2\pi$$

$$x = r \cos \theta$$

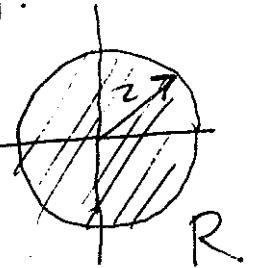
$$y = r \sin \theta$$

$$z = z$$

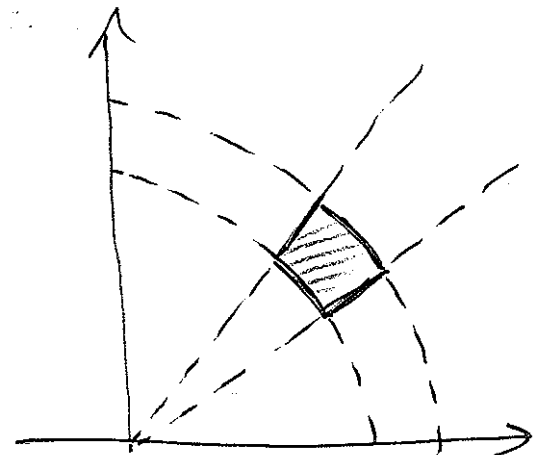
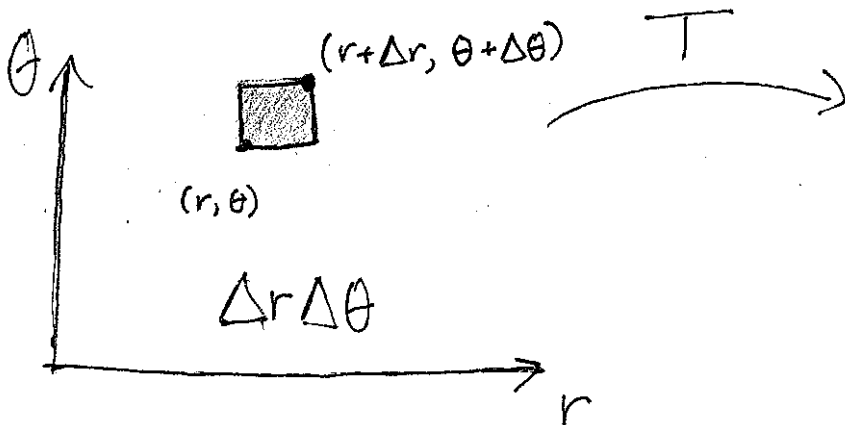


In polar coordinates $dA = r dr d\theta$, e.g.

$$\iint_R xy \, dA = \int_0^{2\pi} \int_0^2 r^2 \cos \theta \sin \theta (r \, dr \, d\theta)$$



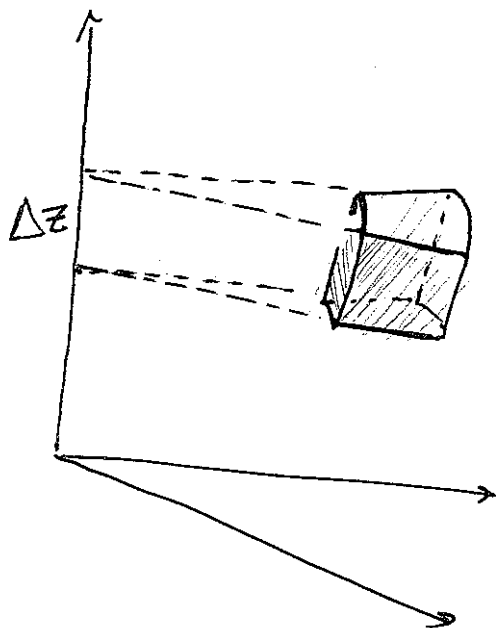
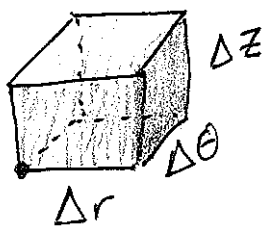
Reason:



$$\text{Area} \approx r \Delta r \Delta \theta$$

$$T(r, \theta) = (r \cos \theta, r \sin \theta)$$

Cylindrical:



At right, volume is

$$\Delta z \cdot \text{Area} = \Delta z (r \Delta r \Delta \theta)$$

$$\Rightarrow dV = r dr d\theta dz$$

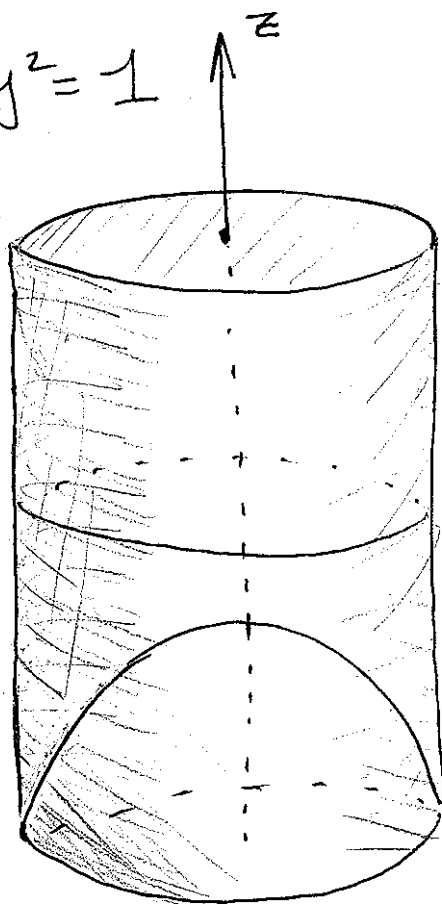
Ex: R inside cylinder $x^2 + y^2 = 1$

below $z = 4$

$$\text{above } z = 1 - x^2 - y^2 \\ = 1 - r^2$$

$$\rho(x, y, z) = \sqrt{x^2 + y^2} = r$$

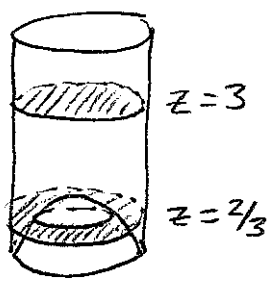
Find total mass.



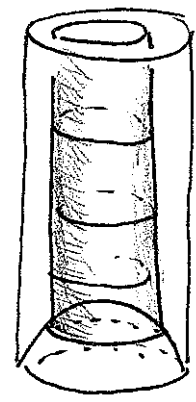
$$\uparrow x^2 + y^2 + z^2 = 1$$

Q: How to slice?

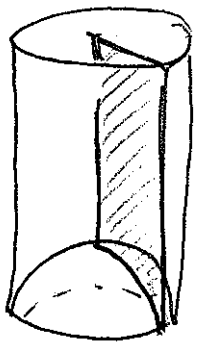
(a) by z:



(b) by r



(c)



by θ

Let's do r, since the book picks θ .

$$\int_0^1 \int_{1-r^2}^4 \int_0^{2\pi} \underbrace{\rho}_{r} \underbrace{dV}_{r d\theta dz dr}$$

$$= \int_0^1 \int_{1-r^2}^4 2\pi r^2 dz dr = \int_0^1 2\pi r^2 z \Big|_{z=1-r^2}^4 dr$$

$$= 2\pi \int_0^1 r^2 (4 - (1-r^2)) dr = 2\pi \int_0^1 (3r^2 + r^4) dr$$

$$= 2\pi \left(r^3 + \frac{1}{5} r^5 \right) \Big|_{r=0}^{r=1} = \frac{12}{5} \pi.$$

Spherical Coordinates:

$$0 \leq \rho$$

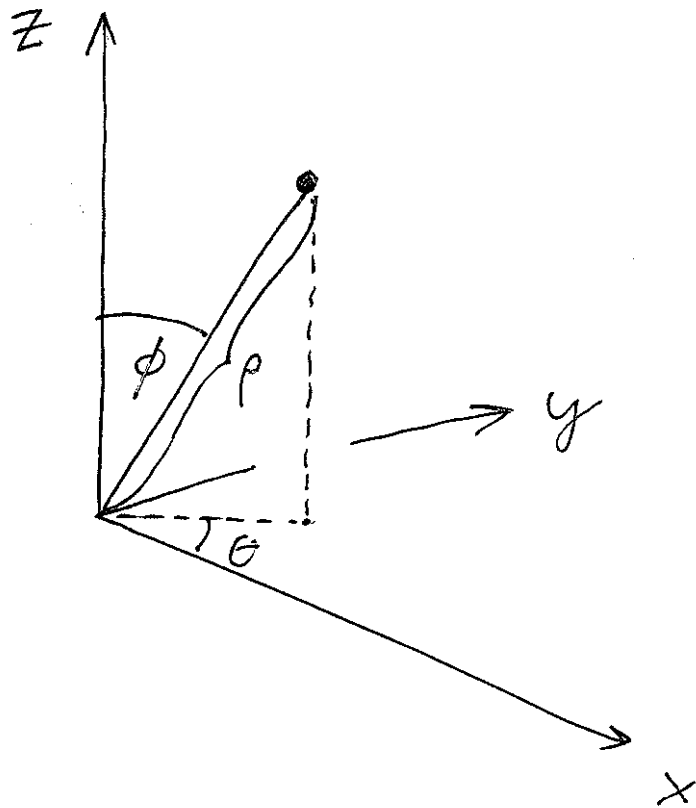
$$0 \leq \theta \leq 2\pi$$

$$0 \leq \phi \leq \pi$$

$$x = \rho \sin \phi \cos \theta$$

$$y = \rho \sin \phi \sin \theta$$

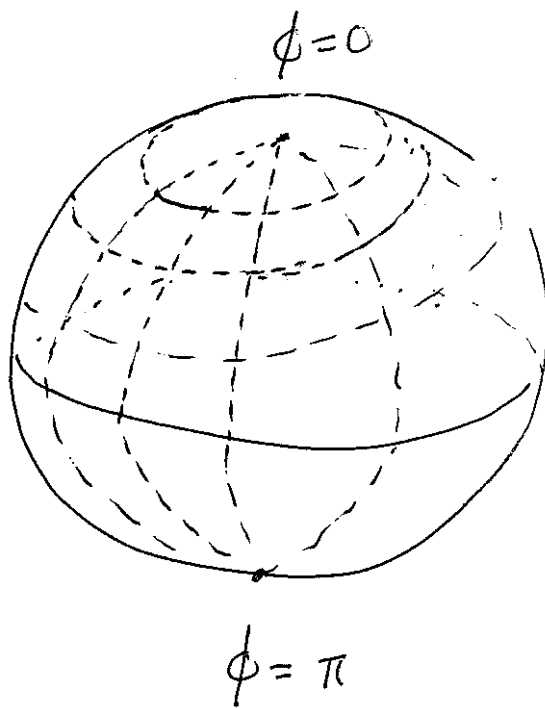
$$z = \rho \cos \phi$$



Q: What is dV here?

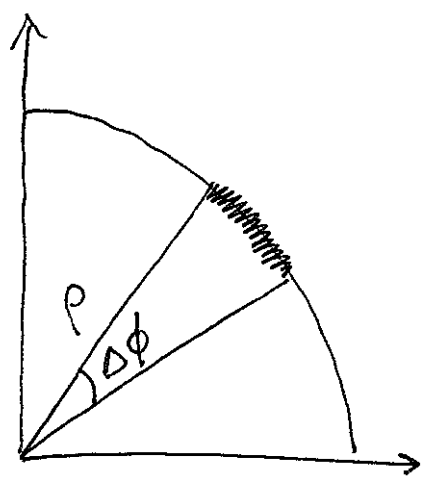
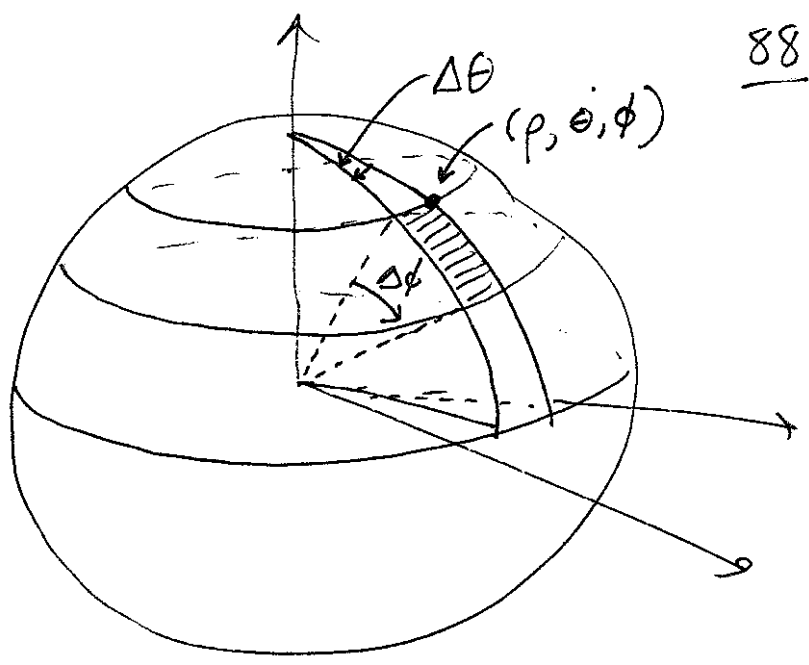
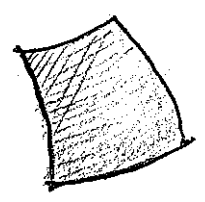
First, (θ, ϕ) give coordinates on the sphere of radius ρ .

(like latitude/longitude)



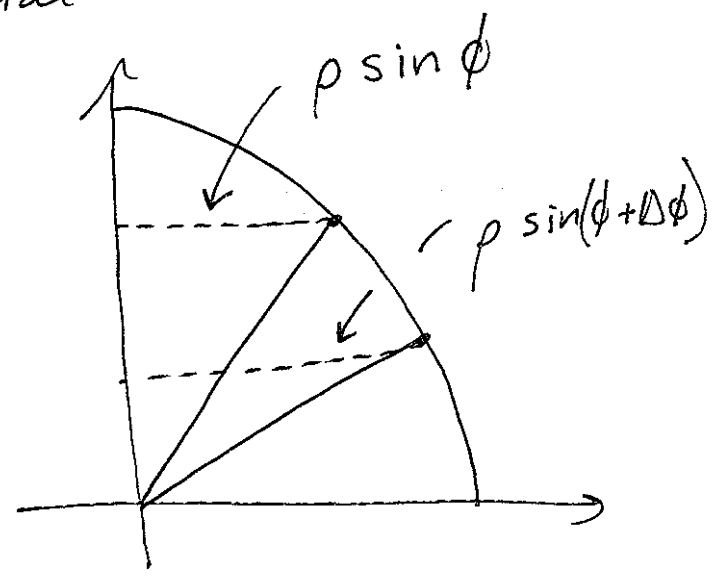
Q1: How much area is the region where θ changes by $\Delta\theta$ and ϕ changes by $\Delta\phi$?

How long are the four sides:



vertical sides = $\rho \Delta\phi$

Horizontal sides



Top = $\Delta\theta \rho \sin \phi$

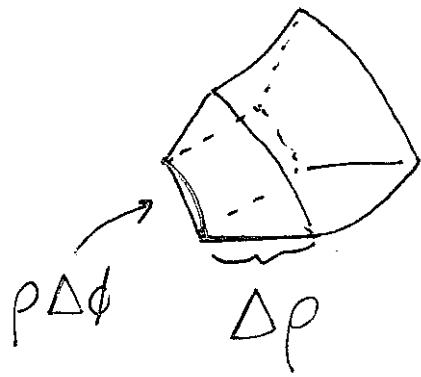
Bottom = $\Delta\theta \rho \sin(\phi + \Delta\phi)$

So, the approximate area is $\rho^2 \Delta\theta \Delta\phi \sin \phi$

(Point: Basically a square, $\sin(\phi + \Delta\phi) \approx \sin \phi + \sin'(\phi) \Delta\phi + \dots$)

So spherical box with sides $\Delta\theta$, $\Delta\phi$, $\Delta\rho$

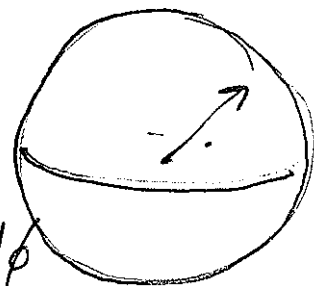
has volume.



$$\rho^2 \sin\phi \Delta\rho \Delta\theta \Delta\phi.$$

Conclusion: $dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi.$

Ex: Volume of sphere of radius 1



$$\iiint_R 1 \, dV = \int_0^\pi \int_0^{2\pi} \int_0^1 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \int_0^\pi \int_0^{2\pi} \left. \frac{\rho^3}{3} \sin\phi \right|_{\rho=0}^1 \, d\theta \, d\phi$$

$$= \int_0^\pi \int_0^{2\pi} \frac{1}{3} \sin\phi \, d\theta \, d\phi = \frac{2}{3} \pi \int_0^\pi \sin\phi \, d\phi$$

$$= \frac{2}{3} \pi (-\cos\phi) \Big|_{\phi=0}^\pi = \frac{2}{3} \pi (---1 - (-1))$$

$$= \frac{4\pi}{3} \quad \text{Same as last time!}$$