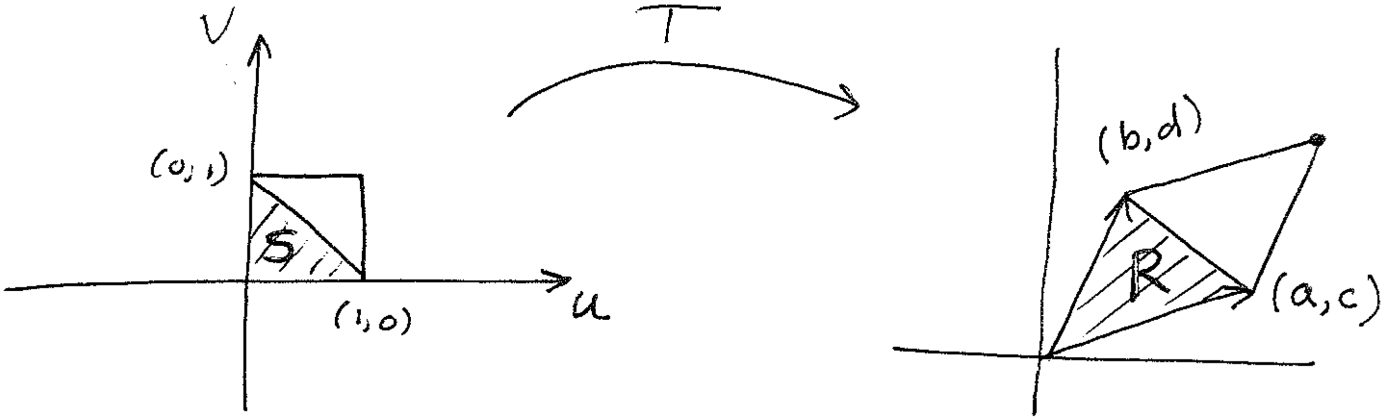
Lecture 28: Changing coordinates II (§15.9)

Next time: Surfaces in $\mathbb{R}^3$ (§16.6)

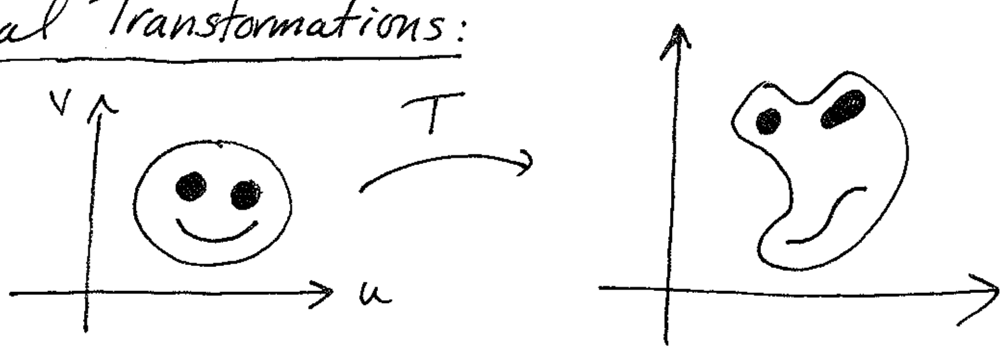
Last time: $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ linear transformation assoc to $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

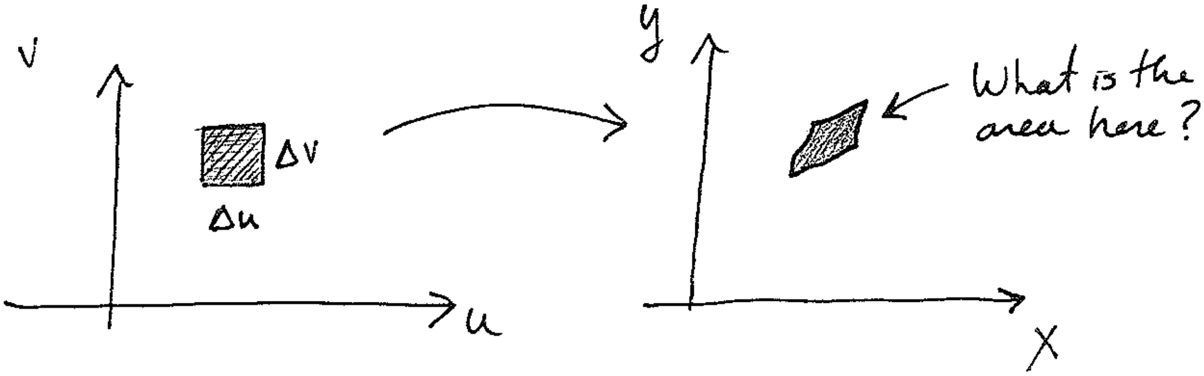
$$T(u,v) = (au+bv,\ cu+dv)$$



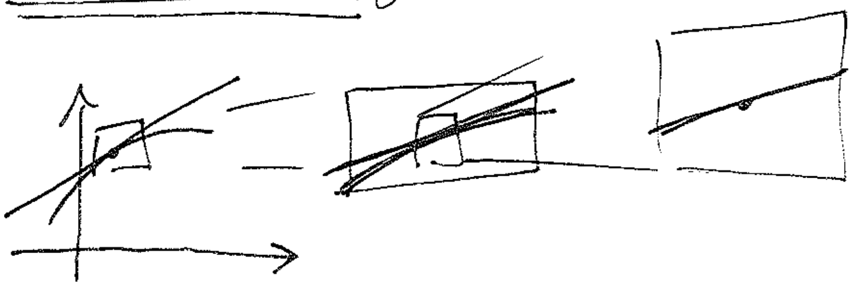
$$\iint_S f(T(u,v)) \begin{vmatrix} a & b \\ c & d \end{vmatrix} \, du\, dv = \iint_R f(x,y)\, dA$$

General Transformations:





[Comes back to a fundamental notion in Calculus: Local linearity. That is, can approx f by linear fns.]



$$f(u+\Delta u) = f(u) + f'(u)\Delta u + \underbrace{E(\Delta u)}_{\text{small}}$$

$$\simeq c + a\Delta u$$

$$g(u+\Delta u, v+\Delta v) = g(u,v) + g_u(u,v)\Delta u + g_v(u,v)\Delta v + \text{Error.}$$

Say that $T: \mathbb{R}^2 \to \mathbb{R}^2$ is <u>differentiable</u> at (u,v) if $T(u,v) = (g(u,v), h(u,v))$ and

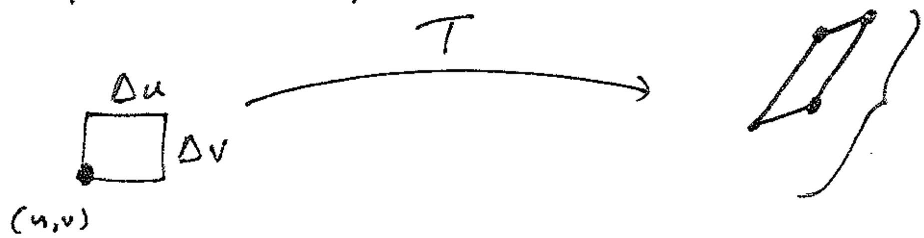
$$T(u+\Delta u, v+\Delta v) = T(u,v) + DT(\Delta u, \Delta v) + E(\Delta u, \Delta v)$$

where DT is the linear trans with matrix
$$J(u,v) = \underbrace{\begin{pmatrix} g_u(u,v) & g_v(u,v) \\ h_u(u,v) & h_v(u,v) \end{pmatrix}}_{\text{Jacobian matrix.}}$$

and $E: \mathbb{R}^2 \to \mathbb{R}^2$ with

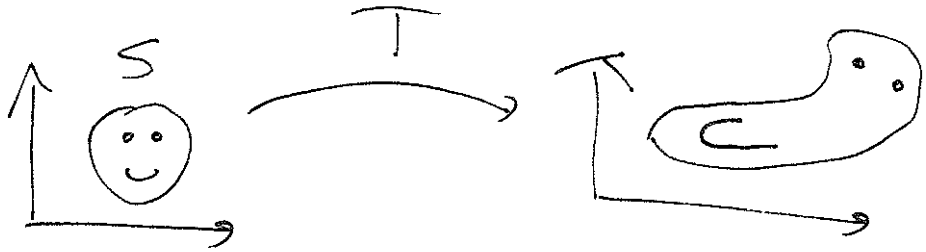
$$\lim_{(\Delta u, \Delta v) \to (0,0)} \frac{|\vec{E}(\Delta u, \Delta v)|}{|(\Delta u, \Delta v)|} = 0.$$

Then if T is diff at (u,v) have



Area $\approx |J(u,v)|\Delta u \Delta v$

So



Then

$$\iint_S f(T(u,v))\underbrace{|\det J|}_{\text{book denotes as } \left|\frac{\partial(x,y)}{\partial(u,v)}\right|}\,du\,dv = \iint_R f(x,y)\,dA$$

Ex: Polar coordinates

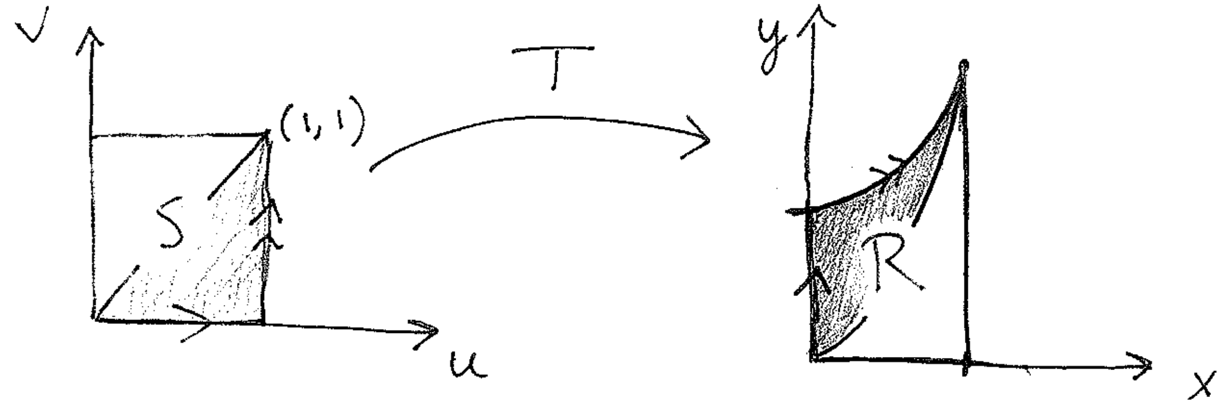
$$T(r,\theta) = (r\cos\theta,\ r\sin\theta)$$

$$J = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

det is $= r\cos^2\theta + r\sin^2\theta = r$

So $dA = r\,dr\,d\theta$, as before.

Ex: (Worksheet) $T(u,v) = (v,\ u(1+v^2))$



$$J = \begin{pmatrix} 0 & 1 \\ 1+v^2 & 2uv \end{pmatrix} \qquad |\det J| = |\underbrace{-1-v^2}_{\text{negative because of the flip.}}|$$

$$= 1+v^2.$$

Compute

$$\iint_R x+y\ dA = \iint_S (v+u(1+v^2))\overbrace{(1+v^2)\,du\,dv}^{dA}$$

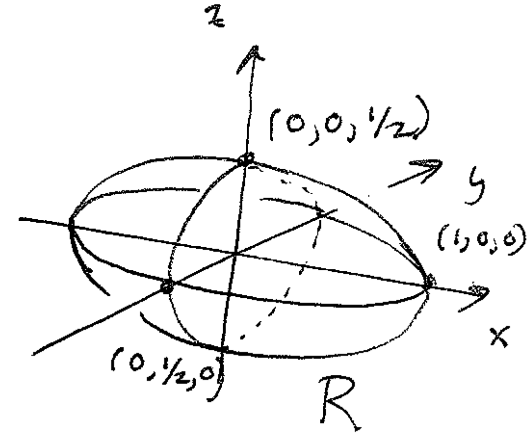
$$= \int_0^1\int_0^1 v+v^3+u(1+2v^2+v^4)\ dv = \int_0^1 v+v^3+\tfrac{1}{2}+v^2+\tfrac{1}{2}v^4\,dv$$

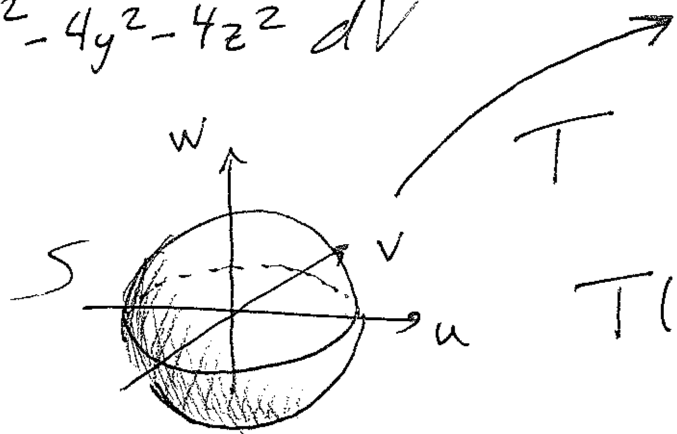
$$= \frac{v^2}{2}+\frac{v^4}{4}+\frac{v}{2}+\frac{v^3}{3}+\frac{1}{10}v^5\Big|_{v=0}^{1} = \frac{101}{60}$$

Changing coordinates in $\mathbb{R}^3$

$$R = \{ \quad x^2 + 4y^2 + 4z^2 \leq 1 \}$$

$$\iiint_R 1 - x^2 - 4y^2 - 4z^2 \, dV$$





$$T(u,v,w) = \left(u, \frac{v}{2}, \frac{w}{2}\right)$$

Since.

$$1 \geq x^2 + 4y^2 + 4z^2 = u^2 + v^2 + w^2$$

T is linear, with matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} = J$

Changes volume by $\det(J) = 1/4$.

So

$$\iiint_R 1 - x^2 - 4y^2 - 4z^2 \, dV$$

$$= \iiint_S (1 - u^2 - v^2 - w^2) \frac{1}{4} \, du\,dv\,dw$$

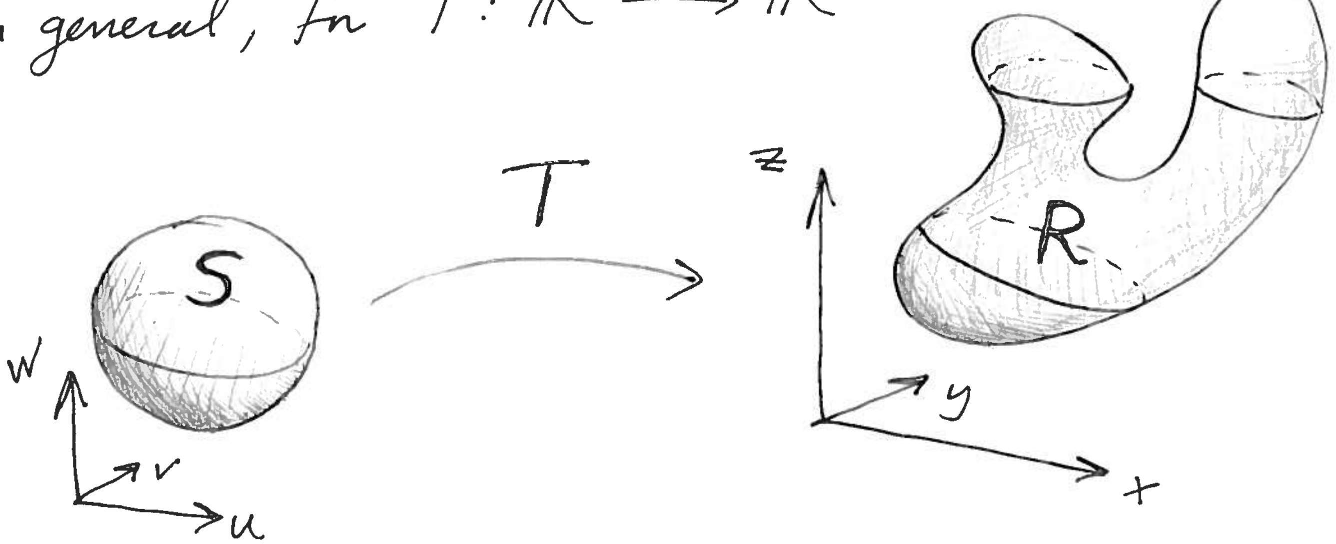
Once more, with feeling.

$$= \int_0^1 \int_0^\pi \int_0^{2\pi} (1-\rho^2)\frac{1}{4}\rho^2 \sin\phi \; d\theta\, d\phi\, d\rho$$

$$= \int_0^1 -\frac{\pi}{2}(1-\rho^2)\rho^2 \cos\phi \Big|_{\phi=0}^{\phi=\pi} d\rho = \int_0^1 \pi(\rho^2-\rho^4)\, d\rho$$

$$= \pi\left(\frac{\rho^3}{3} - \frac{\rho^5}{5}\right)\Big|_{\rho=0}^{\rho=1} = \pi\left(\frac{1}{3}-\frac{1}{5}\right) = \frac{2\pi}{15}$$

In general, for $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$



$$\iiint_S f(T(u,v,w))\,|\det J|\, du\,dv\,dw = \iiint_R f(x,y,z)\, dV$$

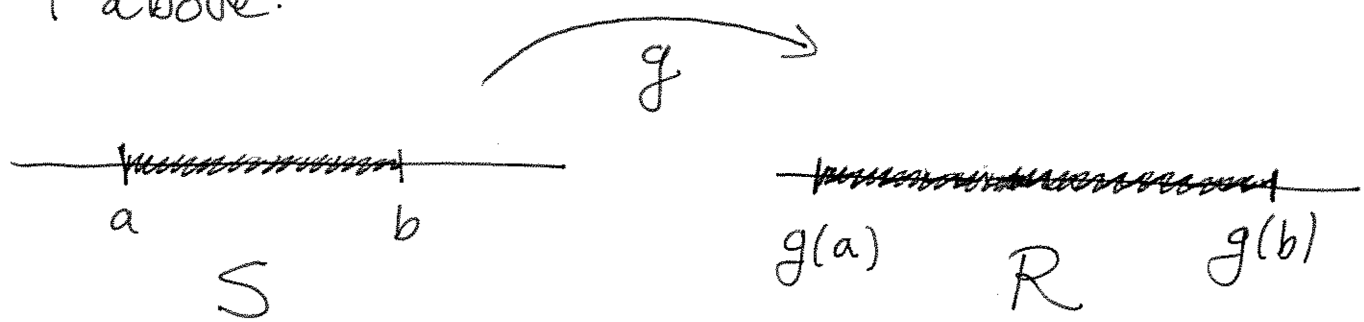
where

$$J = \begin{pmatrix} \frac{\partial T_1}{\partial u} & \frac{\partial T_1}{\partial v} & \frac{\partial T_1}{\partial w} \\ \frac{\partial T_2}{\partial u} & \frac{\partial T_2}{\partial v} & \frac{\partial T_2}{\partial w} \\ \frac{\partial T_3}{\partial u} & \frac{\partial T_3}{\partial v} & \frac{\partial T_3}{\partial w} \end{pmatrix} \quad \text{if } T = (T_1, T_2, T_3).$$

Aside: The 1-variable version of this story is good old integration by subsitition:

$$\int_a^b f(\underbrace{g(t)}_{x}) \underbrace{g'(t)\,dt}_{dx} = \int_{g(a)}^{g(b)} f(x)\,dx$$

for $f, g : \mathbb{R} \to \mathbb{R}$. Here g plays the role of T above:



with $S = [a,b]$ and $R = [g(a), g(b)]$

and the Jacobian "matrix" is just $g'(t)$.

One practical difference is usually our goal with integration by subsitution is to simplify the integrand; with change of coordinates in $\mathbb{R}^2$ and $\mathbb{R}^3$, we are usually trying to simplify the region of integration.