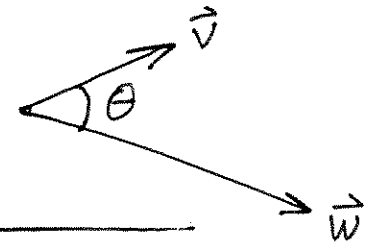
Lecture 3: Projection via the dot product (§12.3). Lines and planes in $\mathbb{R}^3$ (§12.5).

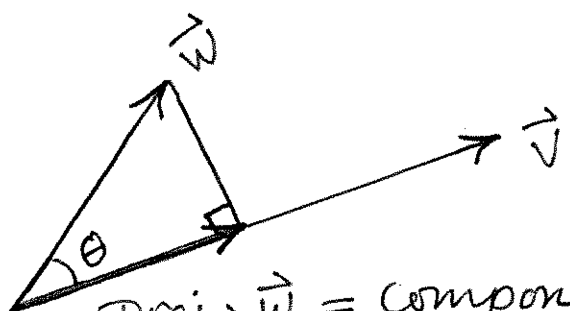
Last time: $\vec{v} = (v_1, v_2, v_3)$ $\quad \vec{w} = (w_1, w_2, w_3)$

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

Key: $\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos\theta$



Projection:



$\text{proj}_{\vec{v}} \vec{w}$ = component of $\vec{w}$ along $\vec{v}$.
= scalar mult. of $\vec{v}$ closest to $\vec{w}$.

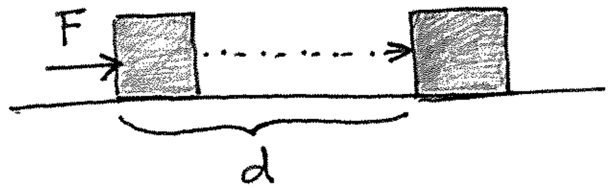
Note: $|\text{proj}_{\vec{v}} \vec{w}| = |\vec{w}| \cos\theta = \vec{v} \cdot \vec{w} / |\vec{v}|$

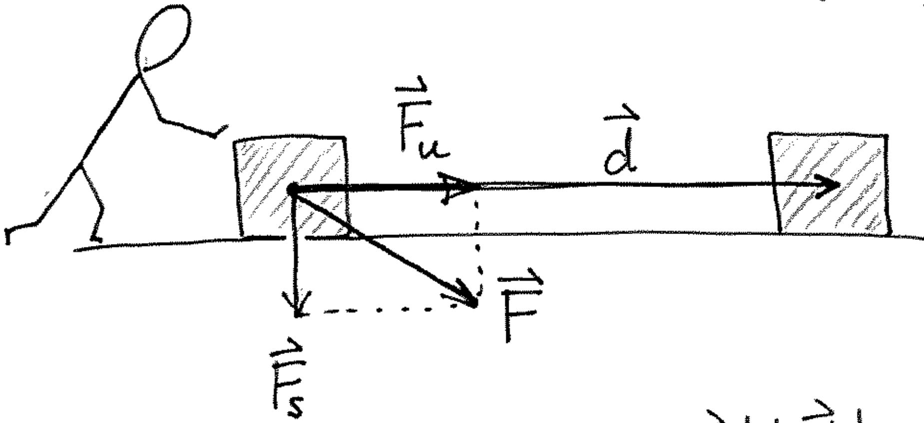
So if $\vec{u}$ = unit vector pointing in same direction as $\vec{v}$ $= \dfrac{\vec{v}}{|\vec{v}|}$, then

$$\text{proj}_{\vec{v}} \vec{w} = |\vec{w}| \cos\theta \, \vec{u} = \left(\frac{\vec{v} \cdot \vec{w}}{|\vec{v}|} \right) \left(\frac{\vec{v}}{|\vec{v}|} \right)$$

$$= \frac{\vec{v} \cdot \vec{w}}{|\vec{v}|^2} \vec{v}$$

Work = (force) × (distance)

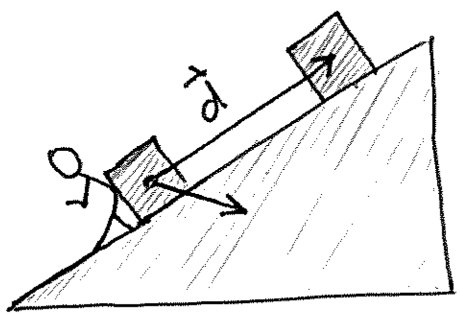




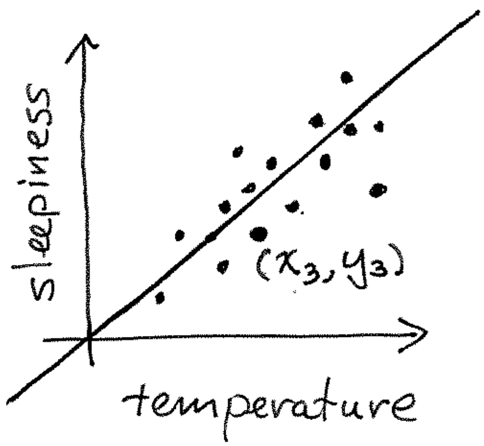
$$W = |\vec{F}_u|\,|\vec{d}| = |\mathrm{proj}_{\vec{d}}\vec{F}|\,|\vec{d}| = \left(\frac{\vec{d}\cdot\vec{F}}{|\vec{d}|}\right)|\vec{d}|$$

$$= \vec{d}\cdot\vec{F}$$

So $\boxed{W = \vec{F}\cdot\vec{d}}$



Regression:



n samples (x_i, y_i)

Roughly, $y_i = c x_i$

What is c?

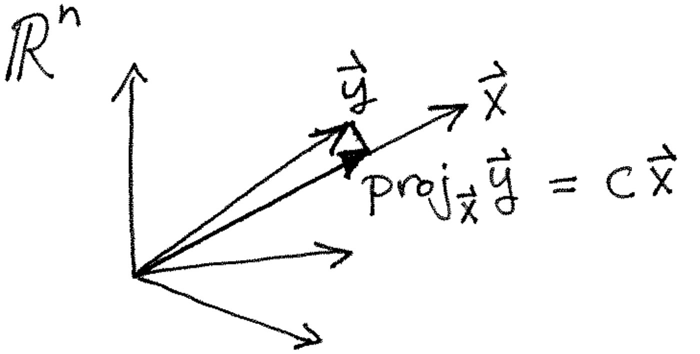
Skip next time.

Consider

$\vec{x} = (x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$

$\vec{y} = (y_1, y_2, \ldots, y_n)$

If $y_i = c x_i$ exactly for all i, then $\vec{y} = c\vec{x}$

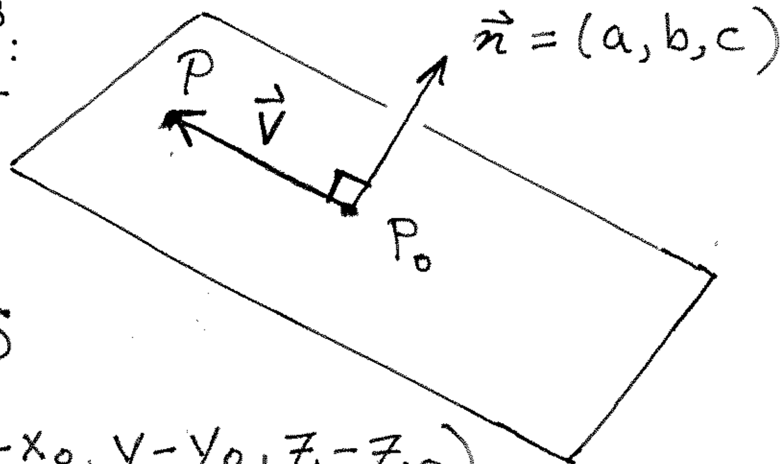


So, the best fit is given by

$$c = \frac{\vec{x}\cdot\vec{y}}{|\vec{x}|^2}$$

[In general, model has more parameters, and projection is onto a plane or a higher dimensional analog. See Math 415/416.]

Planes in $\mathbb{R}^3$:



$$\vec{v} = \overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$$

Can specify by a point $P_0 = (x_0, y_0, z_0)$ and a normal vector $\vec{n}$ that meets the plane at right angles. Another point $P = (x, y, z)$ is in our plane exactly when $\vec{n}$ and $\vec{v}$ are perpindicular (orthogonal), i.e. $\vec{n}\cdot\vec{v} = 0$.

Thus P is in the plane when

$$0 = (a,b,c)\cdot(x-x_0,\ y-y_0,\ z-z_0)$$
$$= a(x-x_0)+b(y-y_0)+c(z-z_0)$$
$$= ax+by+cz+d \quad \text{where } d=-(ax_0+by_0+cz_0)$$

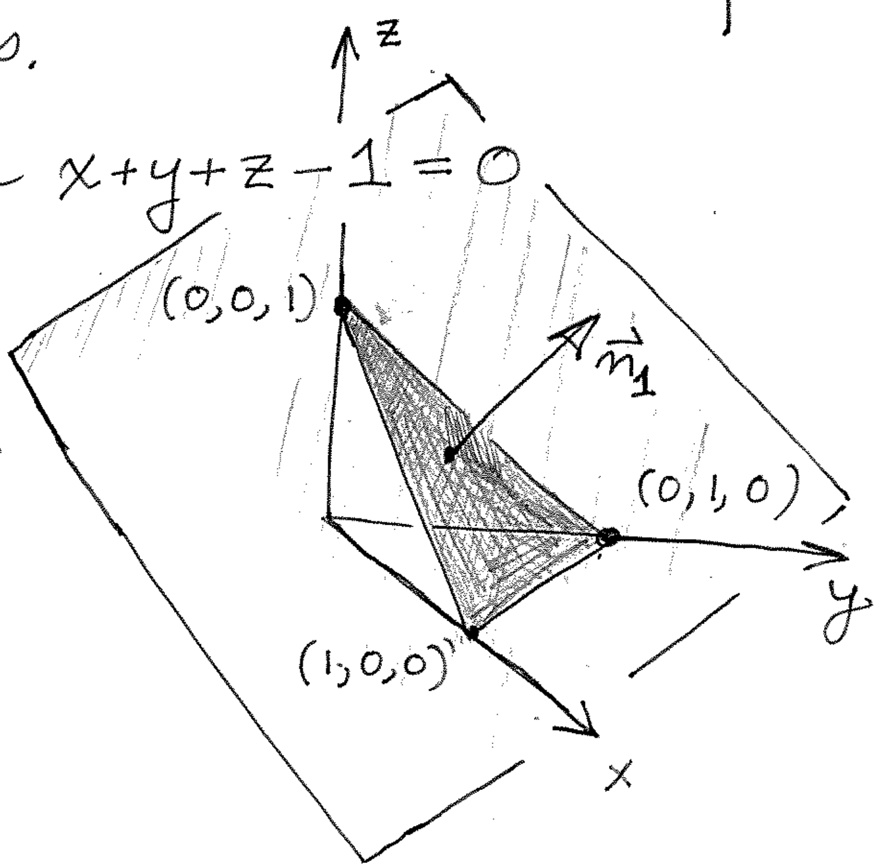
Conversely,

$$ax+by+cz+d=0$$

defines a plane, unless $a=b=c=0$.

[Normal vectors will be key concept later in this course. For now, will use to solve geometric problems about planes.]

<u>Ex</u>: P_1 = Plane given $x+y+z-1=0$

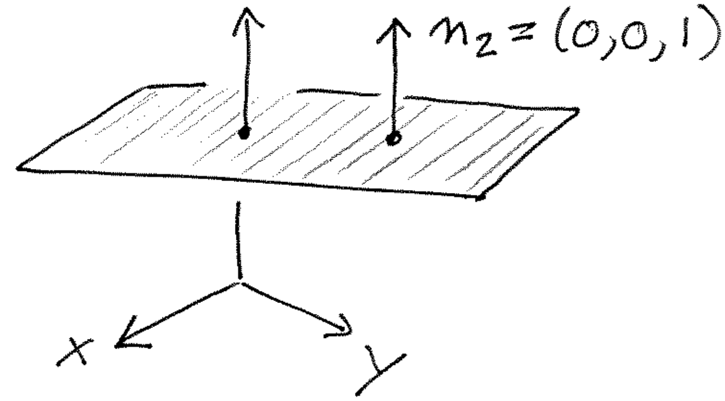


Any plane is determined by 3 pts.

Normal vector is

$$\vec{n}_1 = (1,1,1)$$

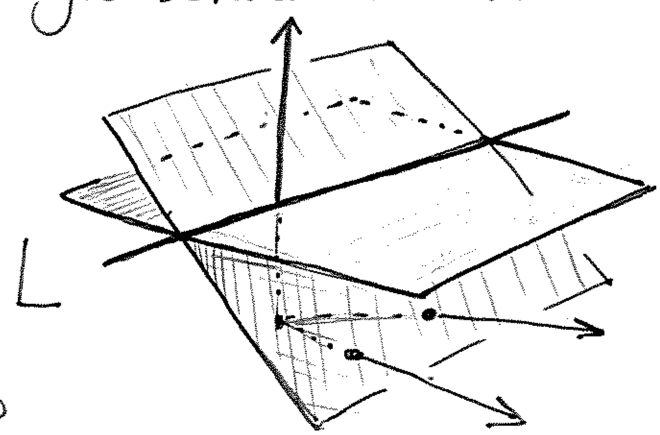
$P_2 = \{z = 1\}$



<u>Q1</u>: What is the intersection of P_1 and P_2

<u>Q2</u>: What is the angle between them?

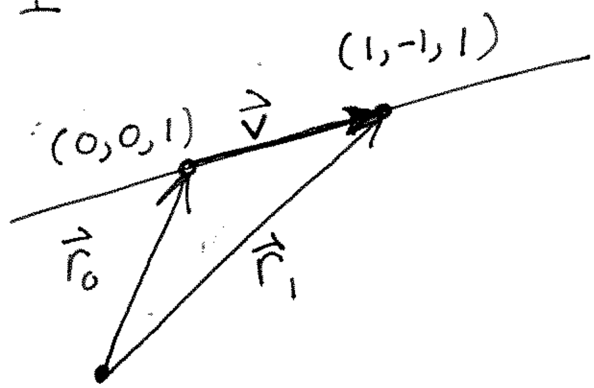
<u>A1</u>: A line:



Points on L satisfy

$$x + y + z = 1 \text{ and } z = 1$$

Two easy solutions:



Get $\vec{v} = \vec{r}_1 - \vec{r}_0$

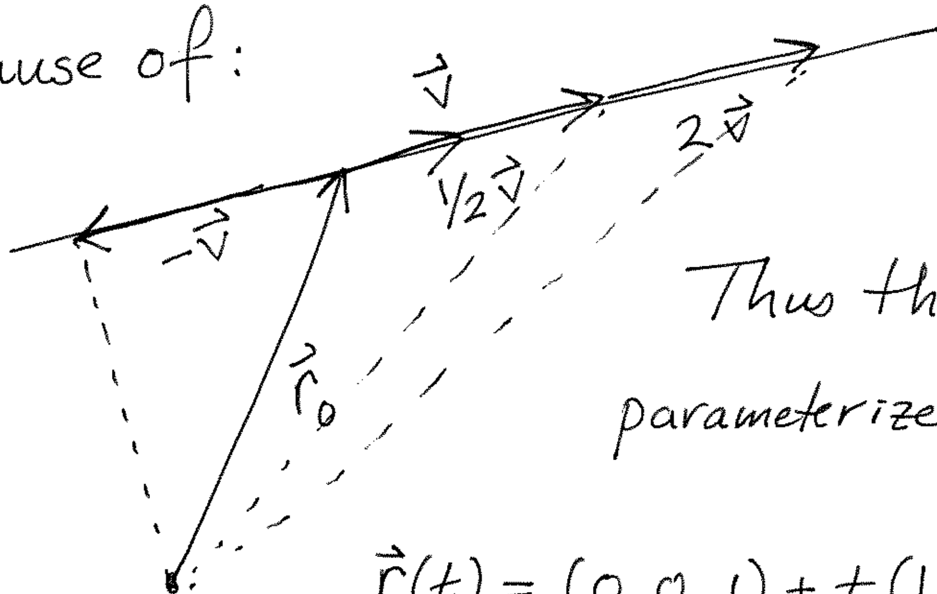
$= (1,-1,1) - (0,0,1)$

$= (1,-1,0)$

Thus every point on L has the form

$$\vec{r}(t) = \vec{r}_0 + t\vec{v}$$

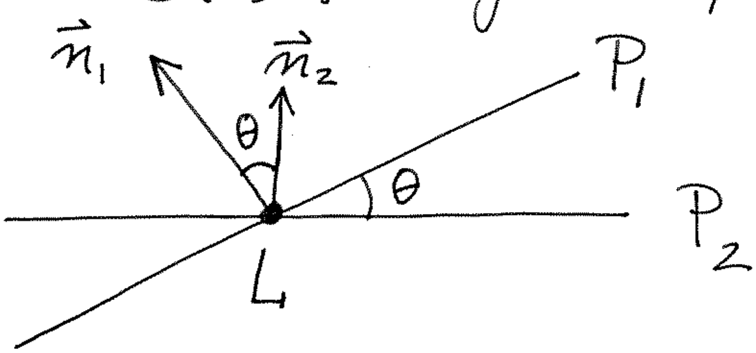
because of:



Thus the line L is parameterized by

$$\vec{r}(t) = (0,0,1) + t(1,-1,0)$$
$$= (0,0,1) + (t,-t,0)$$
$$= (t,-t,0)$$

[This is just like Problem 3(e) from yesterday's worksheet.]



<u>A2</u>: Angle between the planes is the same as the angles between the normals.

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|} = \frac{1}{\sqrt{3}\cdot 1} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta \approx 54.7^\circ$$