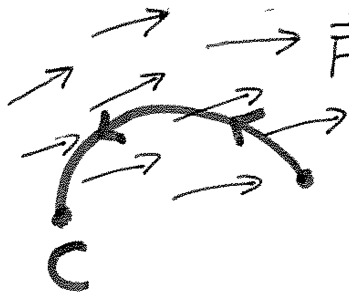
Lecture 31: Green's Theorem (§16.4)

Previously: Some of the new kinds of integrals in this course:

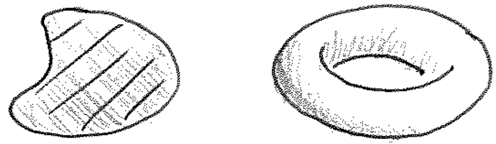
1) Vector fields along curves:



$\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$

$\int_C \vec{F} \cdot d\vec{r}$

2) Functions on 2D regions: $\iint_D f \, dA$



[Onward to new versions of the Fund. Thm of Calc.]

C a closed curve bounding a region D in $\mathbb{R}^2$,

oriented so D is to the left as you go around C.

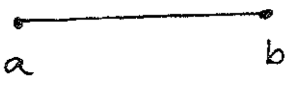
$\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$ a vector field

given by $\vec{F}(x,y) = (P(x,y), Q(x,y))$

Green's Thm:

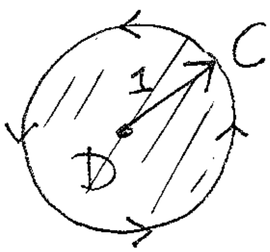
$$\int_C \vec{F} \cdot d\vec{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA$$

[Seems mysterious: how can an integral over the whole region depend only $\vec{F}$ along the boundary curve C.]

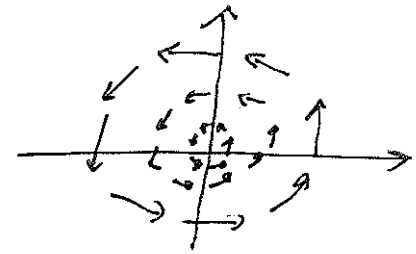
Compare: 

$$f(b) - f(a) = \int_a^b f'(x)\,dx$$

$$f(B) - f(A) = \int_C \nabla f \cdot d\vec{r}$$

Ex: 

$$\vec{F} = \tfrac{1}{2}(-y, x)$$



$$\vec{r}(t) = (\cos t, \sin t) \qquad 0 \le t \le 2\pi$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$$= \int_0^{2\pi} \tfrac{1}{2}(-\sin t, \cos t) \cdot (-\sin t, \cos t)\,dt$$

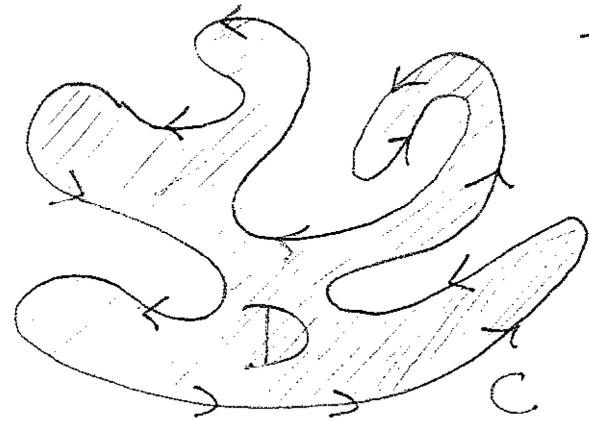
$$= \tfrac{1}{2}\int_0^{2\pi} (-\sin t)^2 + (\cos t)^2\,dt = \tfrac{1}{2}\int_0^{2\pi} 1\,dt = \boxed{\pi}$$

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA = \iint_D \tfrac{1}{2}(1 - (-1))\,dA = \iint_D 1\,dA$$

$$= \text{Area}(D) = \boxed{\pi}$$

Same answer both times!

Special Case: D any region

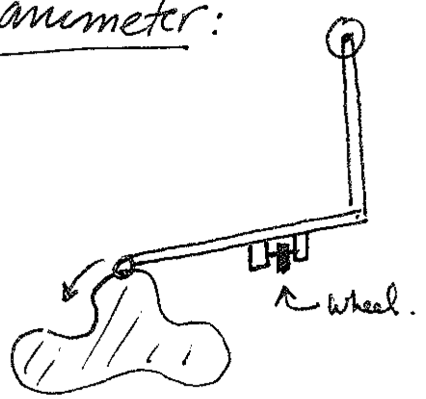


Take $\vec{F} = \frac{1}{2}(-y, x)$

Then

$$\text{Area}(D) = \int_C \vec{F} \cdot d\vec{r}$$

Planimeter:



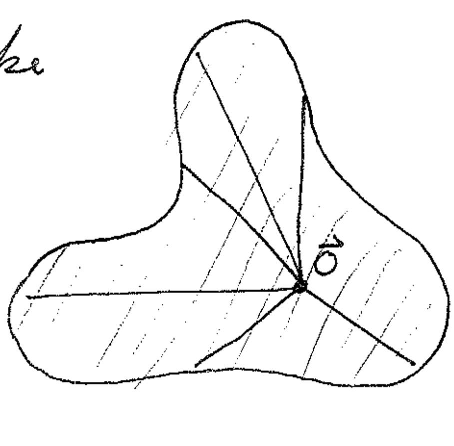
The planimeter is a still-manufactured mechanical device that uses this to measure areas

Perfected in 1854... see web link for details.

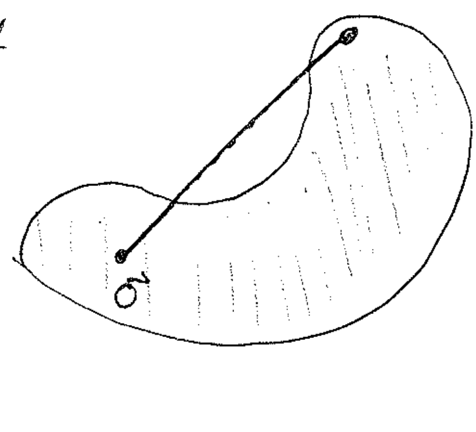
Why Green's Thm works in this special case:

Suppose $\vec{0}$ is in D, which is "star shaped"

like



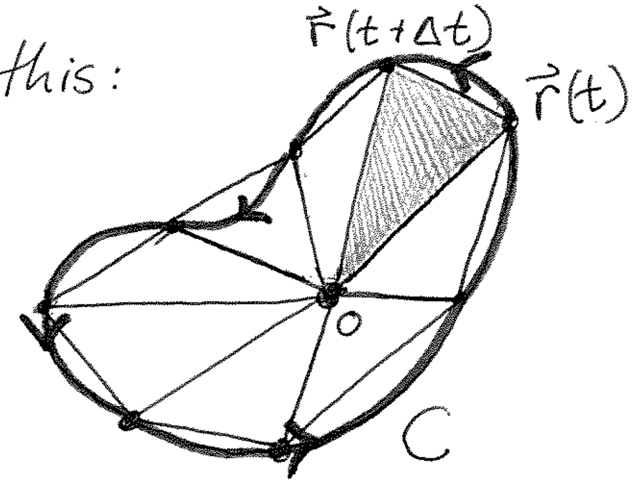
not



[i.e. can see entire room D from $\vec{0}$. Is this room star shaped?]

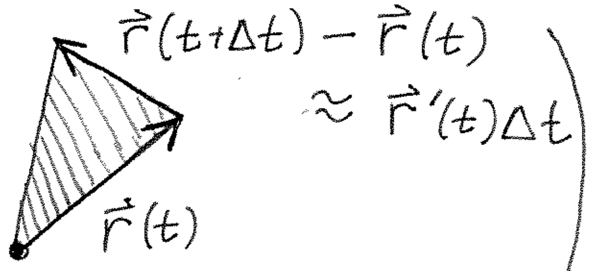
Suppose $\vec{r}:[a,b] \longrightarrow \mathbb{R}^2$ is a parameterization of C.

Use it to estimate Area(D) like this:



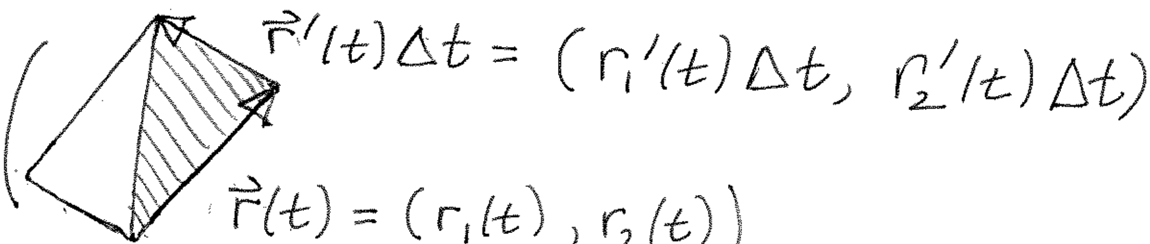
Now

Area (



)

$\approx \frac{1}{2}$ Area (



)

$$= \frac{1}{2}\begin{vmatrix} r_1(t) & r_1'(t)\Delta t \\ r_2(t) & r_2'(t)\Delta t \end{vmatrix} = \frac{1}{2}\big(r_1(t)r_2'(t) - r_2(t)r_1'(t)\big)\Delta t$$

Adding up the areas of such triangles gives a Riemann sum, and so taking $\Delta t \to 0$ we learn:

$$\text{Area}(D) = \int_a^b \frac{1}{2}\big(r_1(t)r_2'(t) - r_2(t)r_1'(t)\big)\,dt$$

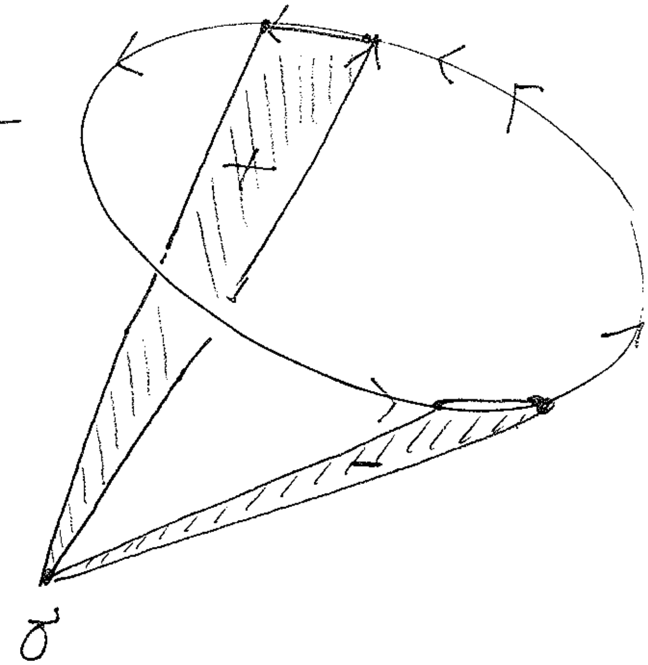
[Pause for questions]

$$= \int_a^b \frac{1}{2}\big(-r_2(t), r_1(t)\big)\cdot\big(r_1'(t), r_2'(t)\big)\,dt$$

$$= \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt = \int_C \vec{F}\cdot d\vec{r}.$$

What if $\vec{0}$ is not in D?

Then the Riemann sum includes both positive and negative contributions. The latter cancel out the portion of the pos. contribs that are too large, and so again



$$\text{Area}(D) = \int_C \vec{F} \cdot d\vec{r}$$

- Also works when D has more than one boundary component:

$$\int_{\partial D} \vec{F} \cdot d\vec{r} = \sum_{\substack{\text{boundary} \\ \text{curves } C_i}} \int_{C_i} \vec{F} \cdot d\vec{r}$$

$$= \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA$$

as long as you orient the C_i so that D is always to the <u>left</u>.