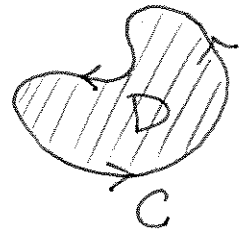


Previously: D region in \mathbb{R}^2 bounded by a curve C
 $\vec{F}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ a vector field with $\vec{F} = (P, Q)$

Green's Thm: $\int_C \vec{F} \cdot d\vec{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA$



Thm A: \vec{F} on a connected open set D in \mathbb{R}^2 is conservative if and only if $\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed curve C in D .

Thm B: If D in \mathbb{R}^2 is simply connected, then

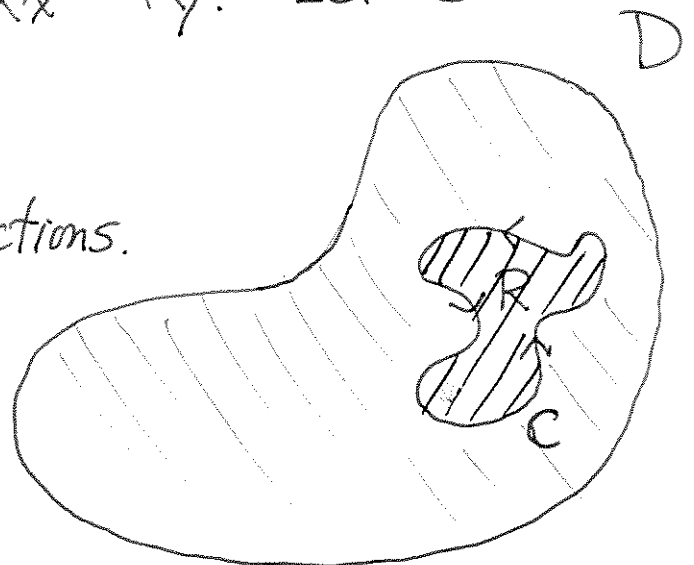
$\vec{F} = (P, Q)$ is conservative if and only if $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$.

[Earlier, explained why Thm A works, and that any conserv. vector field on \mathbb{R}^2 satisfies $Q_x = P_y \dots$]

Reason for Thm B: Suppose D is simply connected and $F = (P, Q)$ satisfies $Q_x = P_y$. Let C be a closed curve in D .

Case 1: C has no self-intersections.

Then C is the boundary of a region R inside D

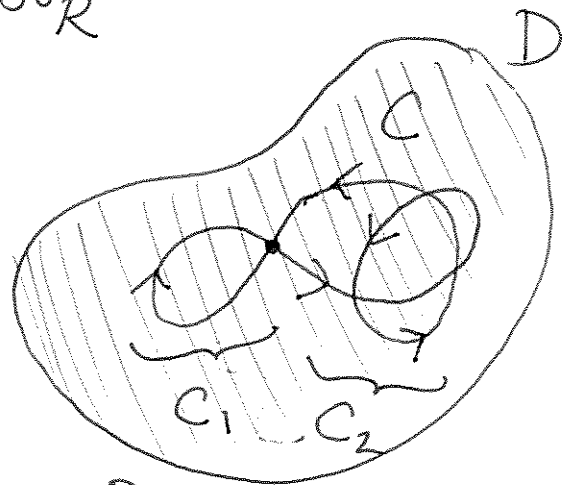


since D "has no holes". Now, by Green's Thm

$$\int_C \vec{F} \cdot d\vec{r} = \iint_R Q_x - P_y \, dA = \iint_R 0 \, dA = 0$$

Case 2: C has self-intersections:

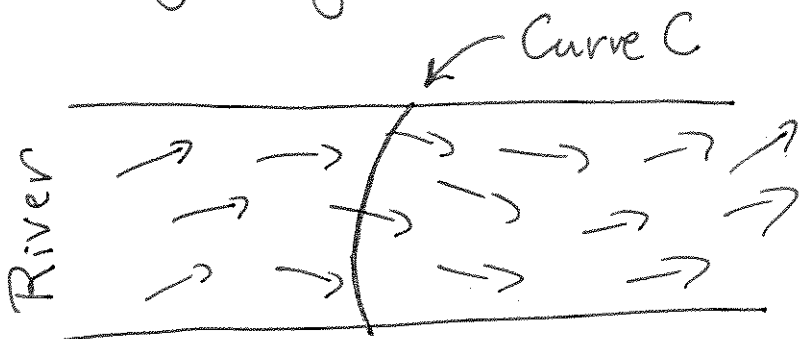
Looking at the first time C meets itself, write $C = C_1 + C_2$ where each C_i is a closed curve and C_1 bounds a region R in D .



Then $\int_{C_1} \vec{F} \cdot d\vec{r} = 0$ as before, and so $\int_C \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$. Repeating, the argument with C_2 eventually shows $\int_C \vec{F} \cdot d\vec{r} = 0$.

Conclusion: Theorem A applies to show \vec{F} is conservative

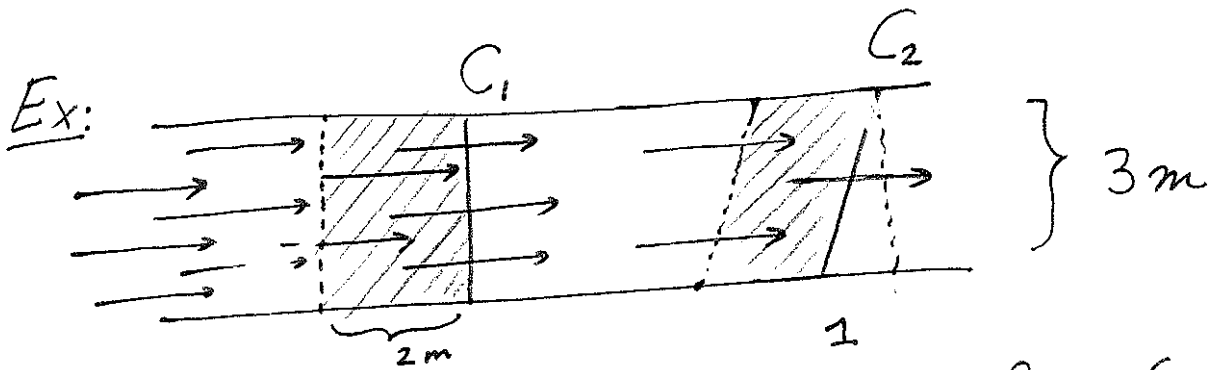
Integrating Vector Fields: Flux in 2D (§16.5)



$$\vec{F}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

Q: What is the rate that the fluid is crossing C ? (= Flux)

Note: Answer will have units (area)/(time) since this is a 2^d-flow.

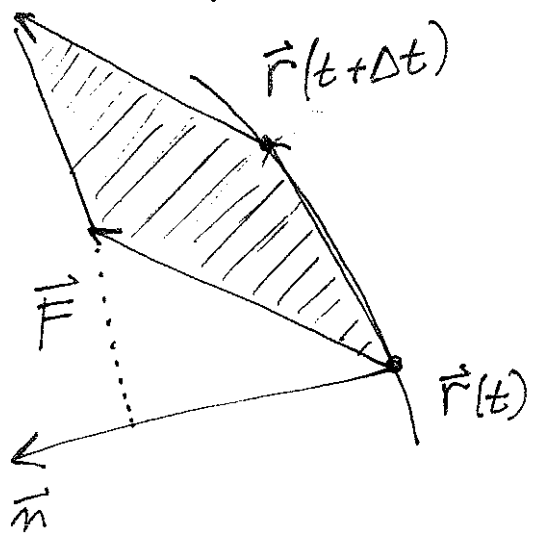


$\vec{F} = (2 \text{ m/s}, 0)$ Flux across $C_1 = 6 \text{ m}^2/\text{s}$

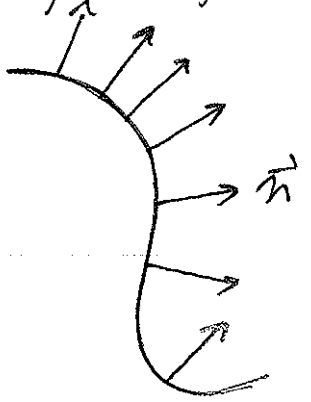
→ Flux across $C_2 = 6 \text{ m}^2/\text{s}$

[Q: Should this be more or less than with C_1 ?]

Closeup of a general curve, where \vec{F} is essentially constant



Let \vec{n} be a unit normal vector field for C .



Area of fluid crossing segment.
 $\approx (\vec{F} \cdot \vec{n}) |\vec{r}'(t)| \Delta t$

function along the curve.

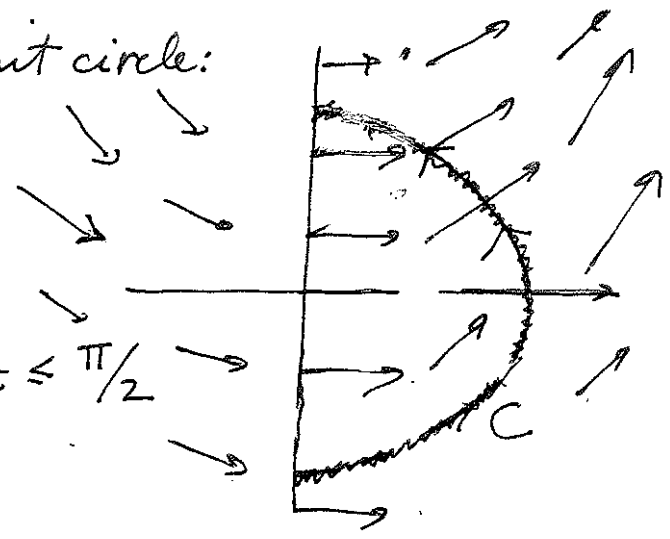
Hence Flux = $\int_C (\vec{F} \cdot \vec{n}) ds$

Ex: $\vec{F} = (1, x)$ and $C = \frac{1}{2}$ unit circle:

Parameterization:

$$\vec{r}(t) = (\cos t, \sin t) \quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$$

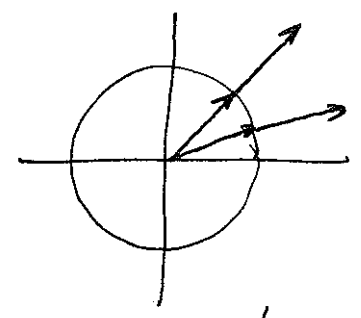
$$\vec{n}(t) = (\cos t, \sin t)$$



Check: Special to this situation!

$$\begin{aligned} \vec{n}(t) \cdot \vec{r}'(t) &= (\cos t, \sin t) \cdot (-\sin t, \cos t) \\ &= 0. \end{aligned}$$

$$|\vec{n}(t)| = 1.$$



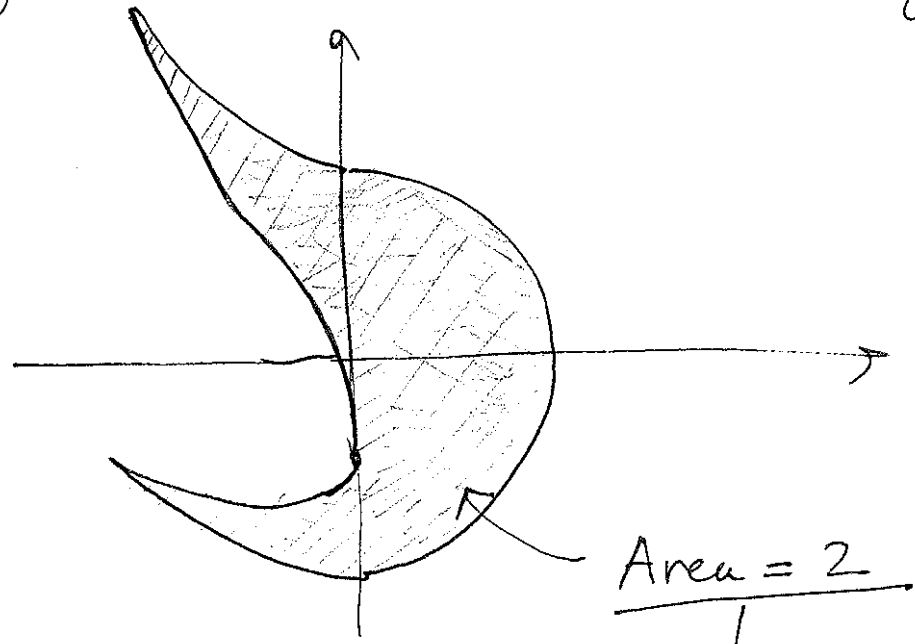
Flux:
$$\int_C (\vec{F} \cdot \vec{n}) ds = \int_{-\pi/2}^{\pi/2} \vec{F}(\vec{r}(t)) \cdot \vec{n}(t) \underbrace{|\vec{r}'(t)|}_{1} dt$$

$$= \int_{-\pi/2}^{\pi/2} (1, \cos t) \cdot (\cos t, \sin t) dt$$

$$= \int_{-\pi/2}^{\pi/2} \cos t + \sin t \cos t dt = 2$$

What does the region of water look like which flows across C in one unit of time?

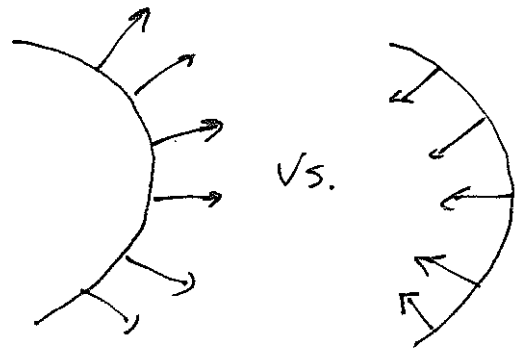
A.



Show animation of this

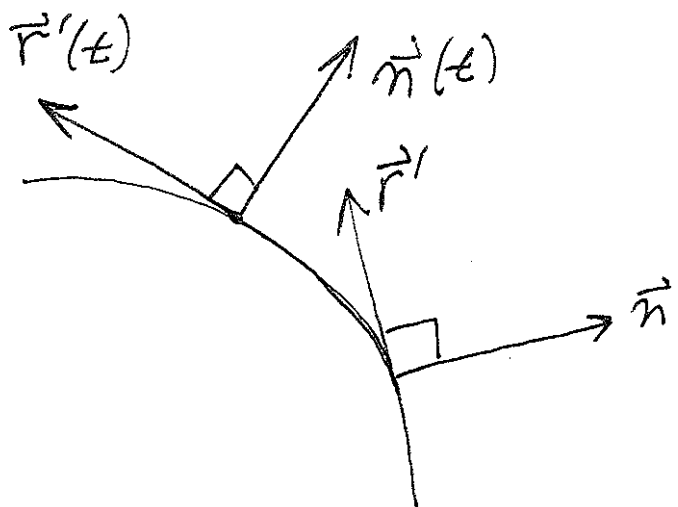
Note to self:
Since $\text{div } \vec{F} = 0$.

Note: When you compute the flux, must choose a direction for \vec{n} :

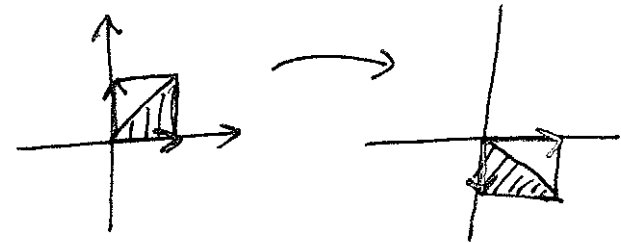


Method for computing \vec{n} :

Assume $\vec{r}: [a, b] \rightarrow C$ has unit speed (e.g. param by arc length).



Relation between:
rotate by 90° $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$



$$T(u, v) = (v, -u)$$

So if $\vec{r}(t) = (r_1(t), r_2(t))$ have

$$\vec{n}(t) = (+r_2'(t), -r_1'(t))$$

Now suppose $\vec{F} = (F_1, F_2)$. Then

units speed
= ds

$$\int_C (\vec{F} \cdot \vec{n}) ds = \int_a^b (F_1(\vec{r}(t)), F_2(\vec{r}(t))) \cdot (r_2'(t), -r_1'(t)) dt$$

$$= \int_a^b \vec{F}_1(\vec{r}(t)) r_2'(t) - \vec{F}_2(\vec{r}(t)) r_1'(t) dt$$

$$= \int_C \vec{G} \cdot d\vec{r} \text{ where } \vec{G} = (-F_2, F_1)$$

$$= \iint_D \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \right) dA$$

What does this measure?

Green's Thm, assuming $C = \partial D$