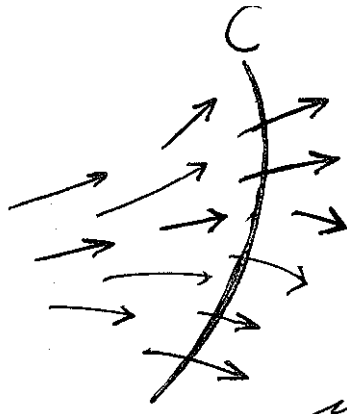


Last time: Flux

$$\vec{F}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$



Flux = rate water is crossing \vec{F}

$$= \int_C (\vec{F} \cdot \vec{n}) ds$$

where \vec{n} is a unit normal vector field along C .

Today, will use the concept of flux to give a different formulation of Green's Theorem, one that will explain why it works, and will also generalize to \mathbb{R}^3

Divergence: $\vec{F}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ a vector field with $\vec{F} = (F_1, F_2)$

$$\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \quad \text{which is [Q.] a function } \mathbb{R}^2 \rightarrow \mathbb{R}$$

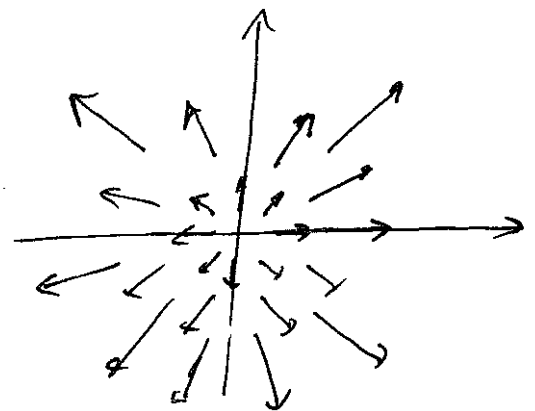
$$\nabla \cdot \vec{F}$$

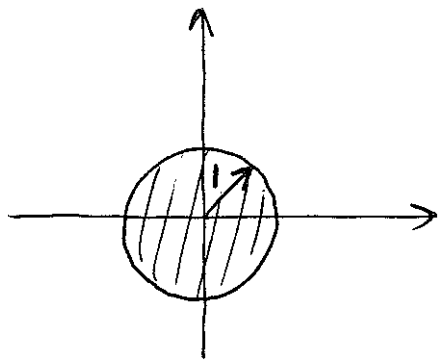
Meaning: [Thinking of \vec{F} as rep. fluid flow:]

$\operatorname{div} \vec{F}$ = rate of expansion of area under the flow.

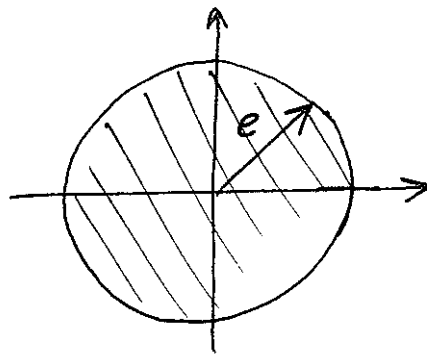
Ex: $\vec{F} = (x, y)$

Suppose we dye the water inside the unit circle green.



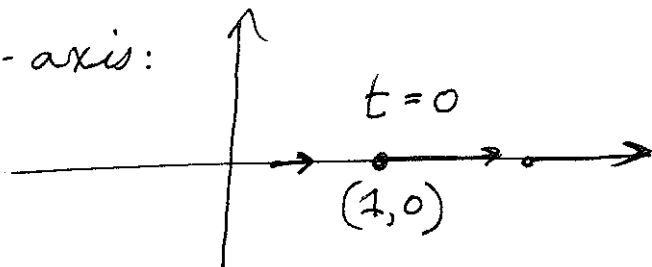


1 unit
of time



Reason: Consider the flow on the x -axis:

$$x'(t) = x(t) \Rightarrow x(t) = e^t$$



In general, the follow of (x_0, y_0) is

given by $\vec{f}(t) = (x_0 e^t, y_0 e^t)$ since $\vec{f}'(t) = \vec{f}(t) = \vec{F}(\vec{f}(t))$

So

$$\frac{\text{Green Area}(t)}{\text{Green Area @ } t=0} = \frac{2\pi(e^t)^2}{2\pi} = e^{2t} \quad \begin{array}{l} \sqrt{\text{rate of increase}} \\ \text{in the} \\ \text{green area.} \end{array}$$

Note

$$\text{div } \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = 2.$$

match!

In real life, how could this happen?

Ⓐ Fluid becomes more/less dense (compressible)

[When modeling airflow over a wing, usually assume $\text{div } \vec{F} = 0$

Ⓑ Fluid is being added somehow.

if speeds are $< \text{Mach } 0.3$]

For any Δt , get a transformation $T_{\Delta t}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

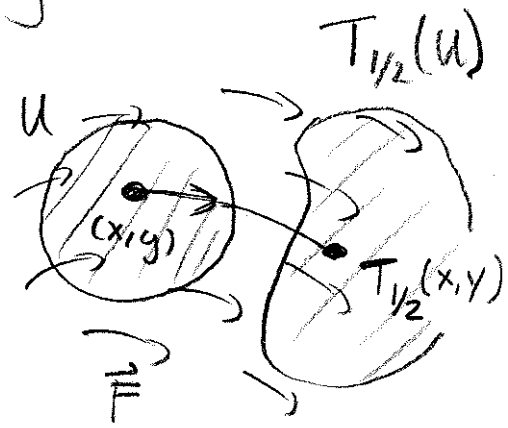
by the rule $T_{\Delta t}(x, y) =$ position of (x, y)
after flowing for time Δt .

For small Δt , setting $\vec{F} = (P, Q)$

have

$$T_{\Delta t}(x, y) \approx (x, y) + \Delta t \vec{F}(x, y)$$

$$= (x + P(x, y) \Delta t, y + Q(x, y) \Delta t)$$



The Jacobian of $T_{\Delta t}$ at (x, y) is about

$$\begin{vmatrix} 1 + P_x(x, y) \Delta t & P_y(x, y) \Delta t \\ Q_x(x, y) \Delta t & 1 + Q_y(x, y) \Delta t \end{vmatrix}$$

$$= 1 + (P_x(x, y) + Q_y(x, y)) \Delta t$$

$$+ (\text{stuff}) \Delta t^2.$$

So the rate of expansion of area is

$$P_x(x, y) + Q_y(x, y) = \text{div } \vec{F}$$

Check units:

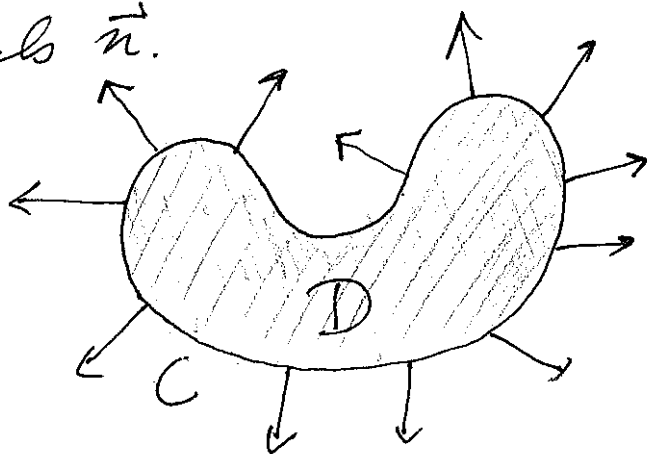
$\vec{F}(x,y)$ has units m/s

$\text{div } \vec{F}$ has units $1/s$ and so $1 + \Delta t (\text{div } \vec{F})$ is dimensionless.

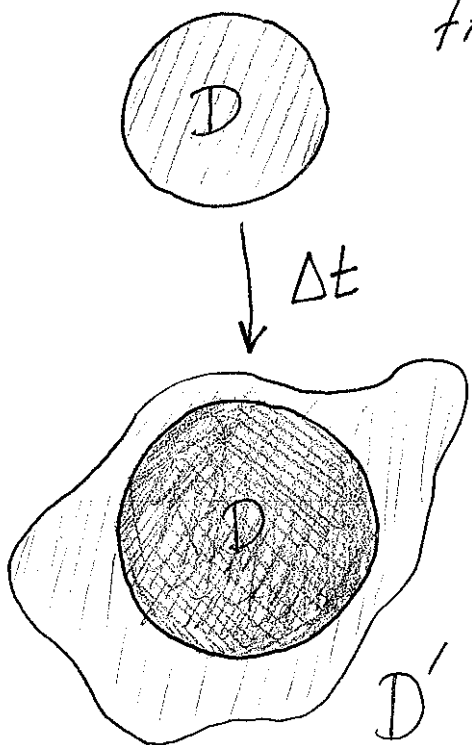
Divergence Thm: D a region in \mathbb{R}^2 bounded by C , with outward unit normals \vec{n} .

Then

$$\underbrace{\int_C \vec{F} \cdot \vec{n} \, ds}_{\text{Flux}} = \iint_D \text{div } \vec{F} \, dA$$



Reason: After time Δt , the fluid in D now fills the region D' . Now



$$\frac{\text{Area}(D')}{\text{Area}(D)} \approx 1 + \Delta t \cdot r$$

where $r = \text{Ave. rate of expansion} = \frac{1}{\text{Area}(D)} \iint_D \text{div } \vec{F} \, dA$

$$\Rightarrow \text{Area}(D') - \text{Area}(D) \approx \Delta t \iint_D \text{div } \vec{F} \, dA$$

Now as the region D is fixed, the change in area can only be accomplished by fluid crossing C . The amount that crosses C in time Δt is $\approx \Delta t \int_C (\vec{F} \cdot \vec{n}) ds$. Thus we must have

$$\int_C \vec{F} \cdot \vec{n} ds = \iint_D \operatorname{div} \vec{F} dA$$

How this relates to Green's Theorem:

If $\vec{F} = (F_1, F_2)$ set $\vec{G} = (-F_2, F_1) = (P, Q)$

Then

$$\int_C (\vec{F} \cdot \vec{n}) ds = \int_C \vec{G} \cdot d\vec{r} \quad (\star)$$

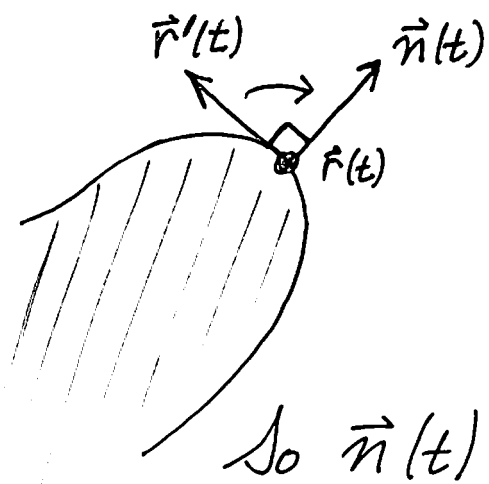
and

$$\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

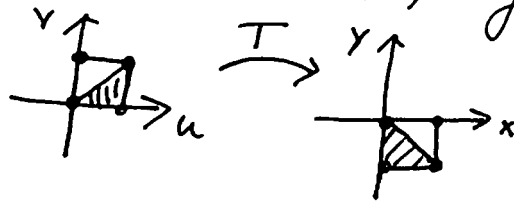
so

$$\iint_D \operatorname{div} \vec{F} dA = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Reason for *: Take a unit-speed param $\vec{r}: [a, b] \rightarrow C$.



Get $\vec{n}(t)$ from $\vec{r}'(t)$ by rotating right



$$T(u, v) = (v, -u)$$

So $\vec{n}(t) = (r_2'(t), -r_1'(t))$ and

$$\int_C \vec{F} \cdot \vec{n} \, ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{n}(t) \, dt$$

$$= \int_a^b F_1(\vec{r}(t)) r_2'(t) - F_2(\vec{r}(t)) r_1'(t) \, dt$$

$$= \int_a^b G(\vec{r}(t)) \cdot \vec{r}'(t) \, dt$$

$$= \int_C G \cdot d\vec{r}$$

Next time: Flux across a surface

