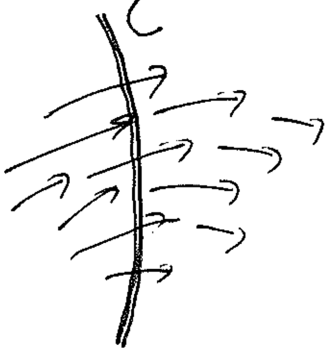
# Lecture 34: Surface integrals of vector fields and the Divergence Theorem in 3D (§16.7 and §16.9)

<u>Previously:</u> $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$

Flux = rate fluid crosses $C$

$$= \int_C (\vec{F} \cdot \vec{n})\, ds$$

where $\vec{n}$ is a unit normal vector field.

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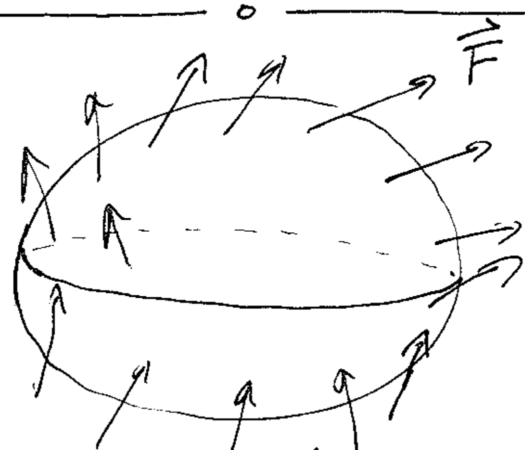
$\vec{F}: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$S$ a surface

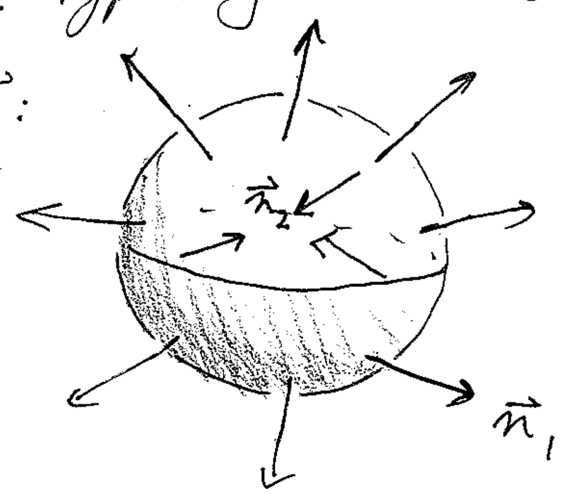


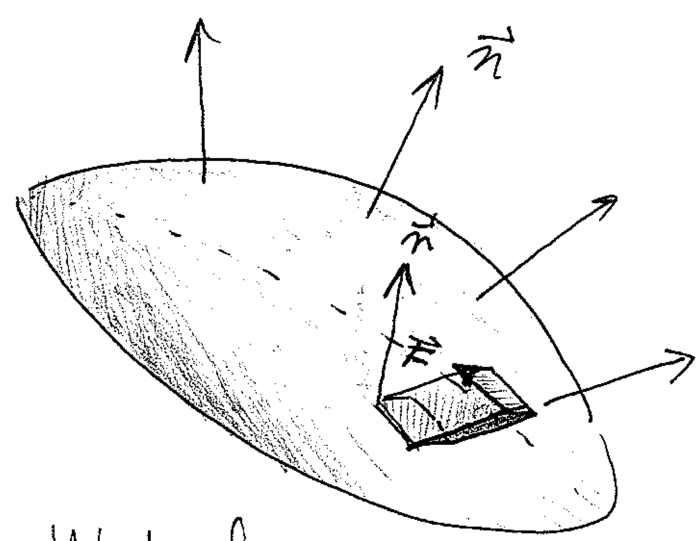

$$\text{Flux} = \iint_S \vec{F} \cdot \vec{n}\, dA = \iint_S \vec{F} \cdot dS$$

↑ Notation

where $\vec{n}$ is a unit normal vector field.

<u>Note:</u> Typically two choices for $\vec{n}$:

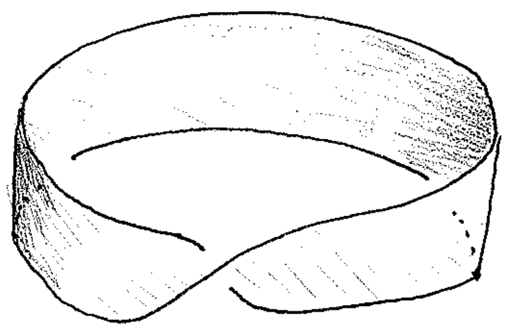


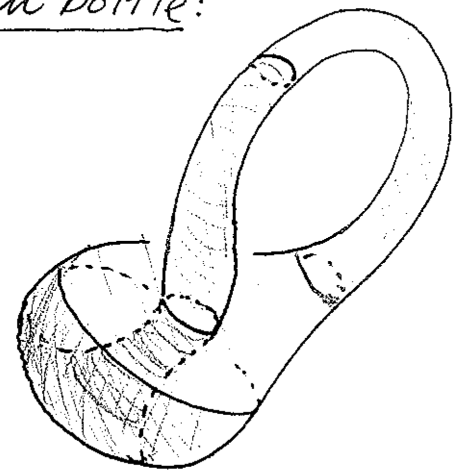

Works for same reason as before.

Some surfaces have no good choice of normal vector, so can't integrate vector fields on these:

Möbius band:



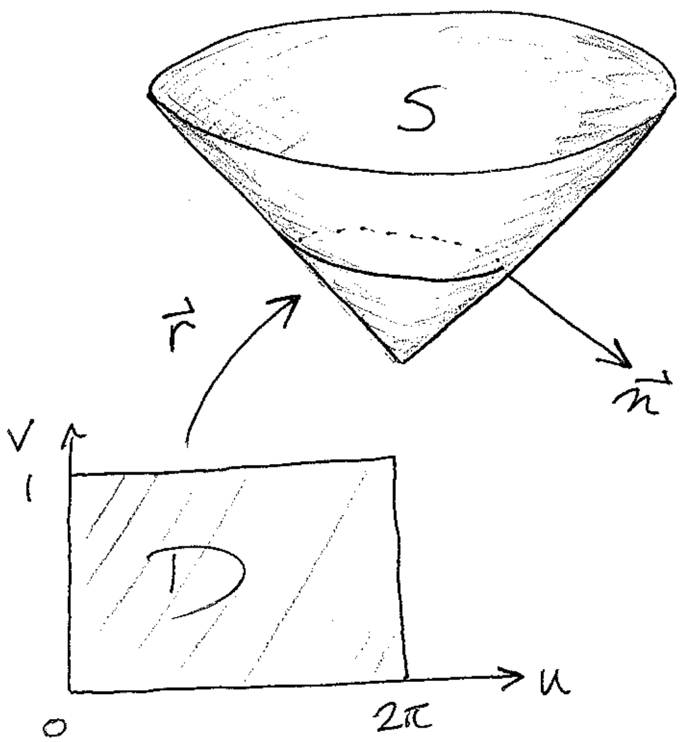
Klein bottle:



[Show physical models.]

Ex: Cone $x^2+y^2=z^2 \quad 0 \le z \le 1$

$\vec{F} = (z, 1, x)$

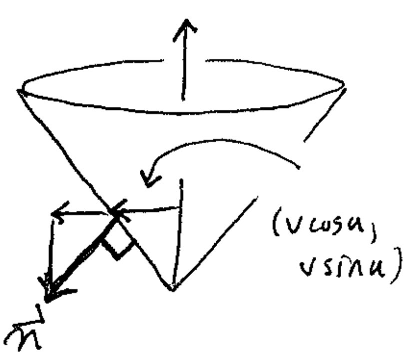



Compute: $\iint_S (\vec{F}\cdot\vec{n})\, dA$

$\vec{r}(u,v) = (v\cos u, v\sin u, v)$

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} = \frac{(v\cos u,\ v\sin u, -v)}{\sqrt{v^2\cos^2 u + v^2 \sin^2 u + v^2}}$$ ← from earlier

$$= \frac{1}{\sqrt{2}}(\cos u, \sin u, -1)$$



$$\underbrace{\vec F(\vec r(u,v))}_{(v,1,v\cos u)}\cdot\vec n = \frac{1}{\sqrt2}\left(v\cos u + \sin u - v\cos u\right)$$

$$= \frac{1}{\sqrt2}\sin u$$

So

$$\iint_S (\vec F\cdot\vec n)\,dA = \int_0^1\int_0^{2\pi}\left(\frac{1}{\sqrt2}\sin u\right)\cdot\overbrace{|\vec r_u\times\vec r_v|}^{\sqrt2 v}\,du\,dv$$

$$= \int_0^1\int_0^{2\pi} v\sin u\,du\,dv = 0$$

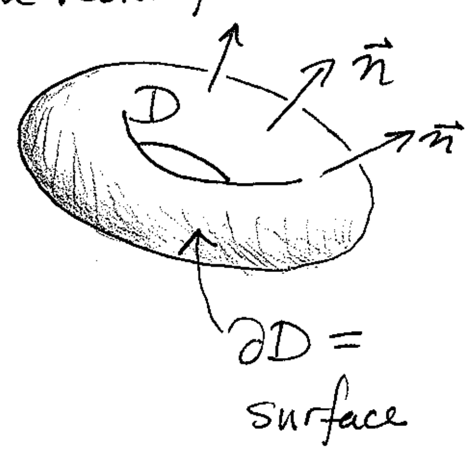
Short cut:

$$\iint_S (\vec F\cdot\vec n)\,dA = \iint_D \vec F\cdot\frac{\vec r_u\times\vec r_v}{\cancel{|\vec r_u\times\vec r_v|}}\cdot\cancel{|\vec r_u\times\vec r_v|}\,du\,dv$$

$$= \iint_D \vec F\cdot(\vec r_u\times\vec r_v)\,du\,dv$$

Divergence Thm: $D$ a region in $\mathbb{R}^3$, $\vec F$ a vector field, $\vec n$ outward pointing normal.

$$\iint_{\partial D}(\vec F\cdot\vec n)\,dA = \iiint_D \operatorname{div}\vec F\,dV$$

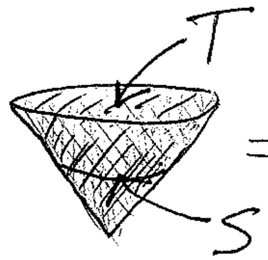


Here, if $\vec{F} = (F_1, F_2, F_3)$ then

$$\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \quad \begin{pmatrix} \text{a fn from} \\ \mathbb{R}^3 \longrightarrow \mathbb{R} \end{pmatrix}$$

Everything can be interpreted as before.

<u>Ex</u>: $D =$

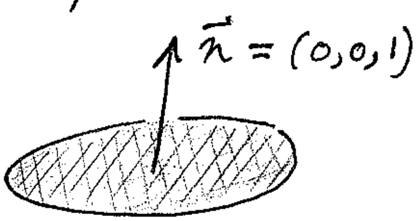



$=$ solid cone $=$ $z^2 \geq x^2 + y^2$, $z \leq 1$

$$\vec{F} = (z, 1, x)$$

$$\iint_{\partial D} (\vec{F} \cdot \vec{n})\, dA = \underbrace{\iint_S (\vec{F} \cdot \vec{n})\, dA}_{0} + \iint_T (\vec{F} \cdot \vec{n})\, dA$$

$$\iint_T (\vec{F} \cdot \vec{n})\, dA = \iint_T x\, dA = 0 \quad \text{by symmetry.}$$




So $\iint_{\partial D} (\vec{F} \cdot \vec{n})\, dA = \boxed{0}$

Compare:

$$\iiint_D \underset{\parallel}{\operatorname{div}} \vec{F}\, dV = \iiint_D 0\, dV = \boxed{0}$$

$$\frac{\partial}{\partial x} z + \frac{\partial}{\partial y} 1 + \frac{\partial}{\partial z} x = 0$$

Application: Heat flow.

$u(x,y,z,t)$ = temperature at $(x,y,z)$ at time $t$.

$$\frac{\partial u}{\partial t} = c\,\underbrace{\Delta}_{\text{Laplacian}} u = c\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

Will now explain why.

Newton's Law of Cooling:

$$\text{Heat flow} = -k\ \mathrm{grad}(u) = \overbrace{-k\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)}^{\text{hot to cold}}$$

Rate heat flows into a region $D$:

$$\iint_{\partial D} (-k\,\mathrm{grad}(u)) \cdot \underbrace{\vec{n}}_{\text{inward normal}}\, dA = \iiint_D k\ \mathrm{div}(\mathrm{grad}\, u)\, dV$$

$$= \iiint_D k\,\Delta u\, dV$$

Ammount of heat energy in $D$:

$$\iiint_D \sigma \rho u\, dV$$

$\left.\begin{array}{l}\sigma \text{ – specific heat} \\ \rho \text{ – mass density}\end{array}\right\}$ constants.

and the rate it is changing is

$$\frac{\partial}{\partial t} \iiint_D \sigma\rho\, u(x,y,z,t)\, dx\,dy\,dz$$

$$= \iiint_D \sigma\rho \frac{\partial u}{\partial t}\, dV$$

$$= \iiint_D k\, \Delta u\, dV$$

As this is true for all regions, must have

$$K \Delta u = \sigma\rho \frac{\partial u}{\partial t} \Rightarrow \frac{\partial u}{\partial t} = \frac{k}{\sigma\rho} \Delta u.$$

<u>Unit check</u>: Heat energy is in J, so flux is $J/s = W$

$k$ in $W/mK$, $\rho$ in $g/m^3$, $\sigma = \frac{J}{gK}$.