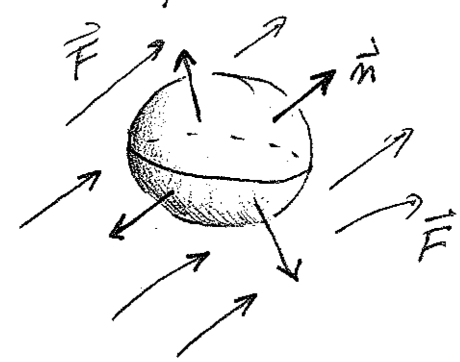
Lecture 36: Stokes Theorem (§16.8), including the definition of the curl (§16.5)

Last time: S surface in $\mathbb{R}^3$, $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ a vector field

$$\text{Flux} = \iint_S (\vec{F} \cdot \vec{n})\, dA$$ where $\vec{n}$ is a unit normal vector field.

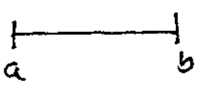


Divergence Thm: D a region in $\mathbb{R}^3$, $\vec{F}: D \to \mathbb{R}^3$ a vector field. Then

$$\iint_{\partial D} (\vec{F} \cdot \vec{n})\, dA = \iiint_D \operatorname{div} \vec{F}\, dV$$

Integral Thms:

1-d:



$$f(b) - f(a) = \int_a^b f'(t)\, dt$$

2-d: Green's Thm

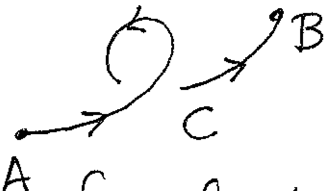
$\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$

$$\int_C \vec{F} \cdot d\vec{r} = \iint_D \frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x}\, dA$$

where $\vec{F} = (P, Q)$.

3-d: Divergence Thm.

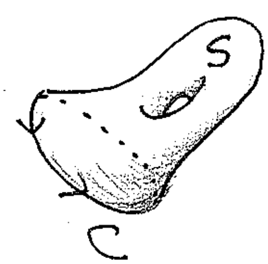
1-d in 3-d:



$$f(B) - f(A) = \int_C \nabla f \cdot d\vec{r}$$

2-d in 3d: Stokes Thm

$\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$



$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\operatorname{curl} \vec{F}) \cdot \vec{n}\, dA$$

[Say something about how these are really the same...]

<u>Curl</u>: $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ a vector field with $F = (F_1, F_2, F_3)$

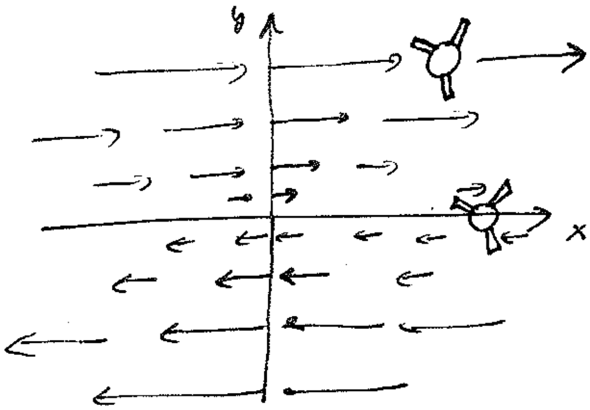
$$\operatorname{curl} \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

<u>Another</u> vector field.

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}$$

<u>Ex</u>: $\vec{F} = (y, 0, 0)$

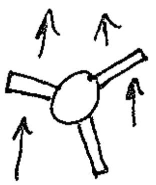
$\operatorname{curl} \vec{F} = (0, 0, -1)$



<u>Q</u>: What does curl measure?

depend only on (x, y).

First, consider $\vec{F} = (\vec{F}_1, \vec{F}_2, 0)$. [which is effectively a vector field on $\mathbb{R}^2$.] Place a small paddle wheel into



the flow. As it moves along with the flow, $|\operatorname{curl} \vec{F}|$ is the rate of rotation

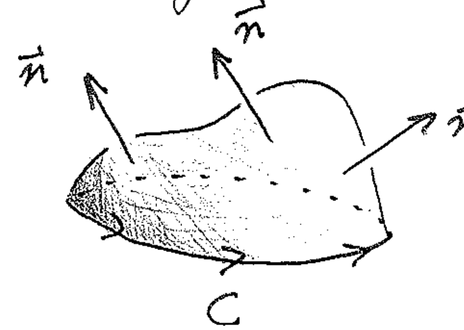
[precisely 2·(angular velocity)]. Also

$$\operatorname{curl} \vec{F} = \left(0, 0, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$ where is > 0 if rotation is anticlockwise.

[Will discuss the general meaning of curl $\vec{F}$ next time.]

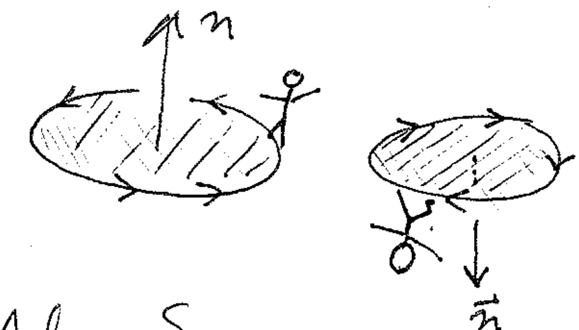
Stokes Thm: S surface in $\mathbb{R}^3$ with boundary curve C.

$\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ a vector field.



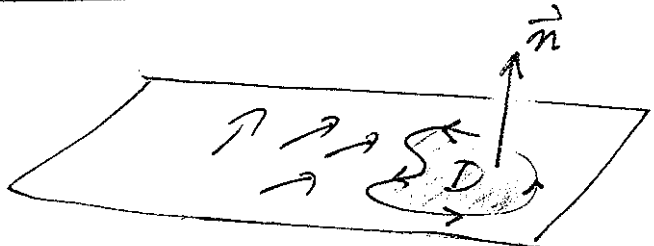
$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\text{curl} \vec{F}) \cdot \vec{n} \, dA$$

C oriented so that S is to your left as you walk around C with head pointing in the normal direction.



Also, S is orientable.

Relation to Green's Thm:



$\vec{F} = (\vec{F}_1, \vec{F}_2, 0)$

$\vec{n} = \vec{k}$

So

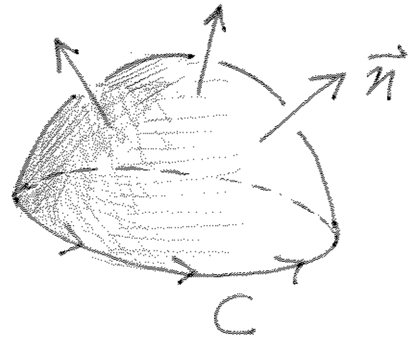
$$(\text{curl} \vec{F}) \cdot \vec{n} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

$$= \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

Example: S = upper unit hemisphere

$\vec{n}$ = outward normal

$\vec{F} = (-y, x, yz)$

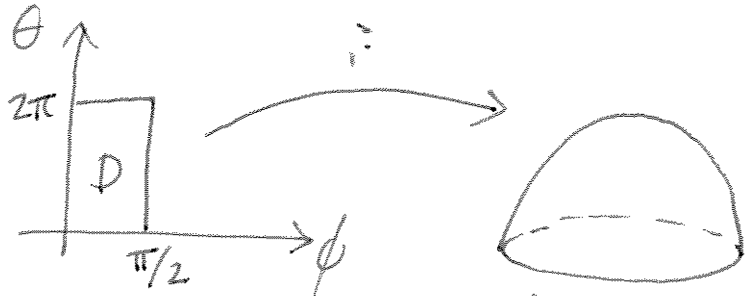


$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & yz \end{vmatrix} = (z, 0, 2)$$

$$\iint_S (\operatorname{curl}\vec{F}) \cdot \vec{n}\, dA = \iint_S (z, 0, 2) \cdot (x, y, z)\, dA$$

$$= \iint_S \underbrace{xz}_{\substack{\text{integrates to } 0 \\ \text{by symmetry}}} + 2z\, dA = \iint_S 2z\, dA$$

Parameterize S:



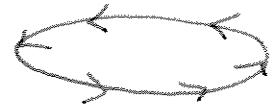
$\vec{r}(\phi, \theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$

$dA = \sin\phi\, d\theta\, d\phi$

$$\iint_S 2z\, dA = \int_0^{2\pi}\int_0^{\pi/2} 2\cos\phi\sin\phi\, d\phi\, d\theta$$

$$= \int_0^{2\pi} \sin^2\phi \Big|_{\phi=0}^{\phi=\pi/2} d\theta = \int_0^{2\pi} 1\, d\theta = \boxed{2\pi}$$

Compare: $\bar{f}(t) = (\cos t, \sin t, 0)$



$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{f}(t))\cdot \vec{r}\,'(t)\,dt$$

$$= \int_0^{2\pi} (-\sin t, \cos t, 0)\cdot(-\sin t, \cos t, 0)\,dt$$

$$= \int_0^{2\pi} \sin^2 t + \cos^2 t\,dt = \int_0^{2\pi} 1\,dt = \boxed{2\pi}\ \checkmark$$

So Stokes' Thm works in this case!

What about the lower hemisphere?

$\pi/2 \le \phi \le \pi$

S'

$$\iint_{S'} (\text{curl}\,\vec{F})\cdot\vec{n}\,dA = \dots = \int_0^{2\pi} \sin^2\phi \Big|_{\phi=\pi/2}^{\phi=\pi} d\theta$$

$$= \int_0^{2\pi} -1\,d\theta = -2\pi.$$

<u>But</u>: C is also the boundary of S', so shouldn't we get $\int_C \vec{F}\cdot d\vec{r} = 2\pi$?

<u>Solution</u>: With outward normal, C gets oriented the other way by S'