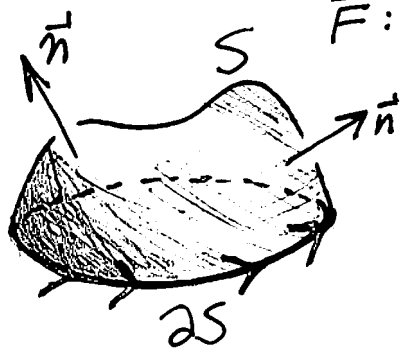


Last time: Stokes' Thm:  $S$  surface in  $\mathbb{R}^3$

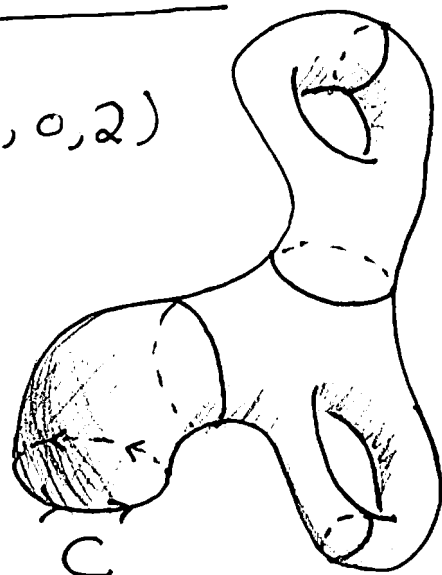
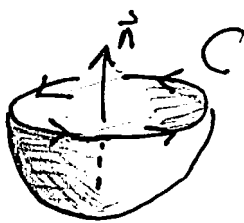
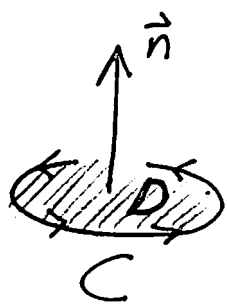
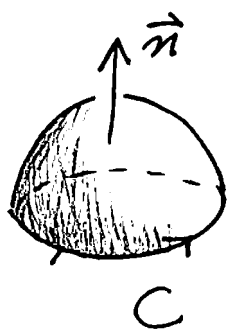
$\vec{F}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  vector field. Then

$$\int_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S (\text{curl } \vec{F}) \cdot \vec{n} \, dA$$



Ex:  $\vec{F} = (-y, x, yz)$

$\text{curl } \vec{F} = (z, 0, 2)$



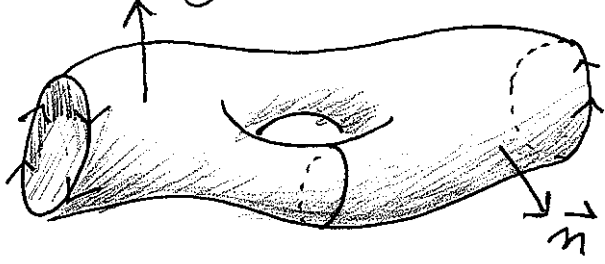
$$\iint_S (\text{curl } \vec{F}) \cdot \vec{n} \, dA = 2\pi = \int_C \vec{F} \cdot d\vec{r} \text{ for all of these!}$$

[ Takes some getting used to, is really just Green's Thm / 2<sup>d</sup> Divergence Thm in disguise... ]

Check the easy one:  $\iint_D (\text{curl } \vec{F}) \cdot \vec{n} \, dA$

$$= \iint_D (z, 0, 2) \cdot (0, 0, 1) \, dA = \iint_D 2 \, dA = 2 \text{Area}(D) = 2\pi$$

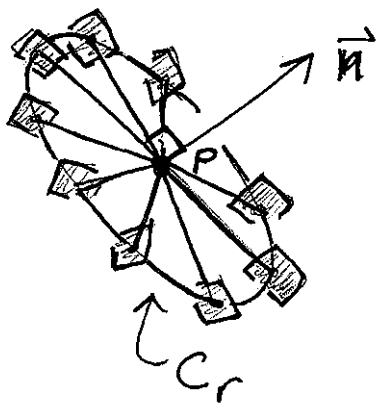
Note: Also works when  $S$  has several boundary components, [provided they are oriented correctly.]



Understanding Curl: Consider a small paddle wheel at  $P$ , of radius  $r$ .

$\vec{F}$  = fluid flow

Q: How fast does it spin?

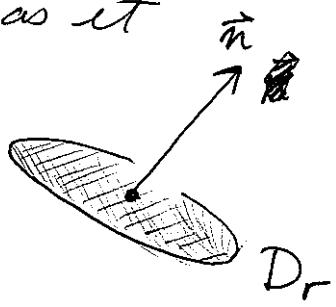


A:  $\omega = \frac{1}{2\pi r^2} \int_{C_r} \vec{F} \cdot d\vec{r}$

units =  $\frac{\text{radians}}{\text{time}}$

Plausible: (a) Want tangential component of  $\vec{F}$  as it hits the paddles.

(b) Looks like an average (almost).



Stokes says:

$$\omega = \frac{1}{2\pi r^2} \iint_{D_r} (\text{curl } \vec{F}) \cdot \vec{n} \, dA$$

$$= \frac{1}{2} \left( \frac{1}{\text{Area}(D_r)} \iint_{D_r} (\text{curl } \vec{F}) \cdot \vec{n} \, dA \right)$$

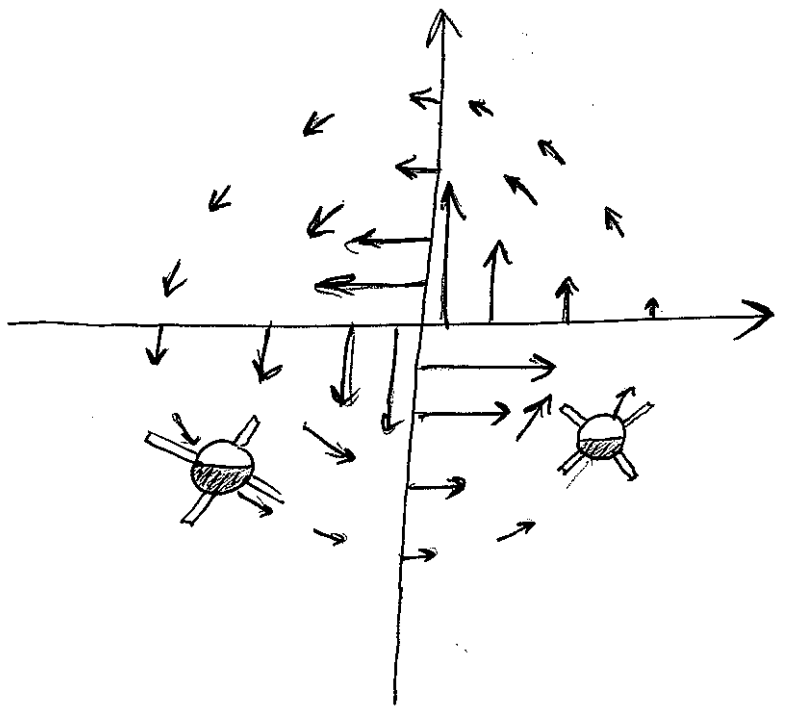
(3)

Taking  $r \rightarrow 0$  get:  $\omega = \frac{1}{2} (\text{curl } \vec{F}) \cdot \vec{n}$

So the rate of rotation is fastest the direction of  $\text{curl } \vec{F}(p)$  and then  $\omega = \frac{1}{2} |\text{curl } \vec{F}|$ .

Note: A vector field where  $\text{curl } \vec{F} = \vec{0}$  everywhere are called irrotational.

Ex:  $\vec{F} = \frac{1}{x^2+y^2} (-y, x, 0)$  has  $\text{curl } \vec{F} = \vec{0}$  except at  $(0,0)$  where it's not defined.



Experimentally, a draining tub is an irrotational flow!

Conservative Vector Fields. Recall  $\vec{F}: \mathbb{R}^n \rightarrow \mathbb{R}^n$  is conservative if  $\vec{F} = \nabla f$  for some  $f: \mathbb{R}^n \rightarrow \mathbb{R}$ .

Thm B: Suppose  $\vec{F} = (P, Q)$  is a vector field on an open simply-connected region  $D$  in  $\mathbb{R}^2$ . Then  $\vec{F}$  is conservative if and only if  $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$  on  $D$ .

[Want this kind of easy to use test for  $\mathbb{R}^3$ .]

Thm B': A vector field  $\vec{F}$  defined on all of  $\mathbb{R}^3$  is conservative if and only if  $\text{curl } \vec{F} = \vec{0}$  everywhere.

Note: Example on last page has  $\text{curl } \vec{F} = \vec{0}$  but is not conservative because of:

Thm A: A vector field  $\vec{F}$  on an open connected region  $R$  in  $\mathbb{R}^n$  is conservative if and only if  $\int_C \vec{F} \cdot d\vec{r} = 0$  for every closed curve  $C$  in  $R$ .

Compatible with Thm B' since  $\vec{F}$  is not defined at  $(0, 0, 0)$ .

Idea behind Thm B', part I. Suppose

$\vec{F}$  is conservative, with  $\vec{F} = \nabla f$  for some  $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ . Then

$$\text{curl } \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \left( \underbrace{\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}}_{=0}, 0, 0 \right)$$

and so  $\text{curl } \vec{F} = \vec{0}$  everywhere.

Next time: I'll explain why  $\text{curl } \vec{F} = \vec{0}$  means  $\vec{F}$  is conservative.