

Previously on Math 241...

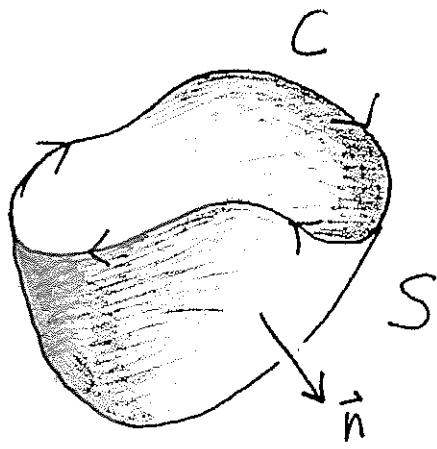
A vector field  $\vec{F}$  is conservative when  $\vec{F} = \nabla f$  for some  $f: \mathbb{R}^n \rightarrow \mathbb{R}$ .

Thm A: A vector field on an open connected region  $R$  in  $\mathbb{R}^n$  is conservative if and only if  $\int_C \vec{F} \cdot d\vec{r} = 0$  for every closed curve  $C$  in  $R$ .

Thm B': A vector field  $\vec{F}$  on all of  $\mathbb{R}^3$  is conservative if and only if  $\text{curl } \vec{F} = \vec{0}$  everywhere.

[Explained part of this last time, will rest now.]

Other idea behind Thm B': Suppose  $\text{curl } \vec{F} = \vec{0}$  and  $C$  is a closed curve in  $\mathbb{R}^3$ . [Need  $\int_C \vec{F} \cdot d\vec{r} = 0$  so that Thm A applies.] Suppose  $S$  is an orientable surface  $S$  with  $\partial S = C$ .



By Stokes:

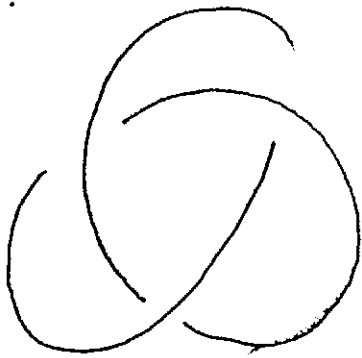
$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\text{curl } \vec{F}) \cdot \vec{n} \, dA$$

$$= \iint_S \vec{0} \cdot \vec{n} \, dA = 0.$$

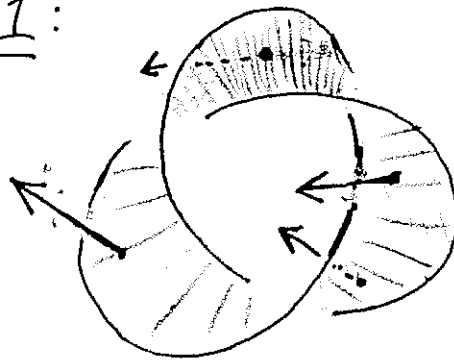
Q: Is every C the boundary of some such S?

If yes, then  $\int_C \vec{F} \cdot d\vec{r} = 0$  for all closed C and so Thm A gives that  $\vec{F}$  is conservative.

Ex:



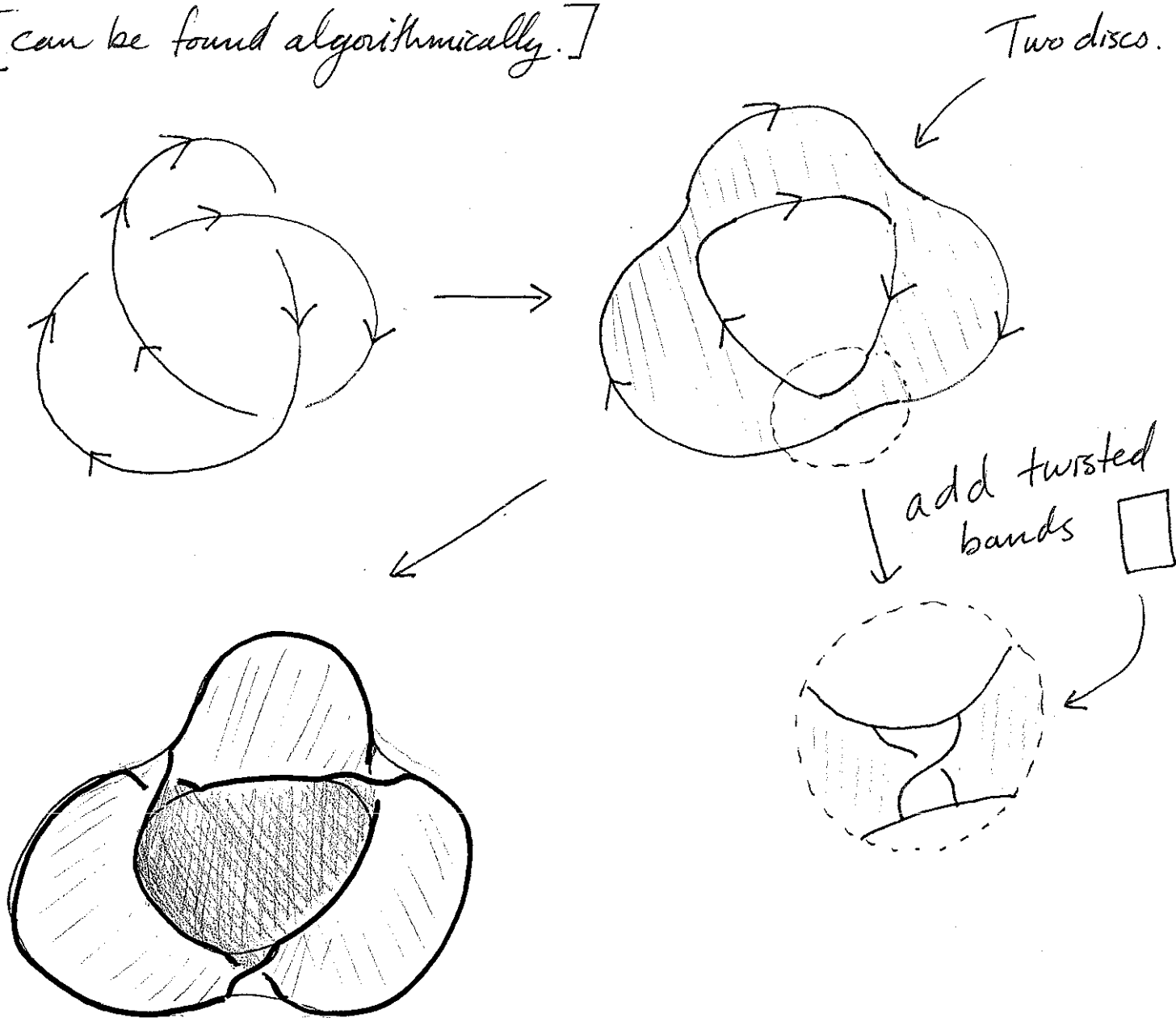
Try 1:



Problem: not orientable.

Fact: There is always such a surface.

In fact, there is always such a surface  
[can be found algorithmically.]

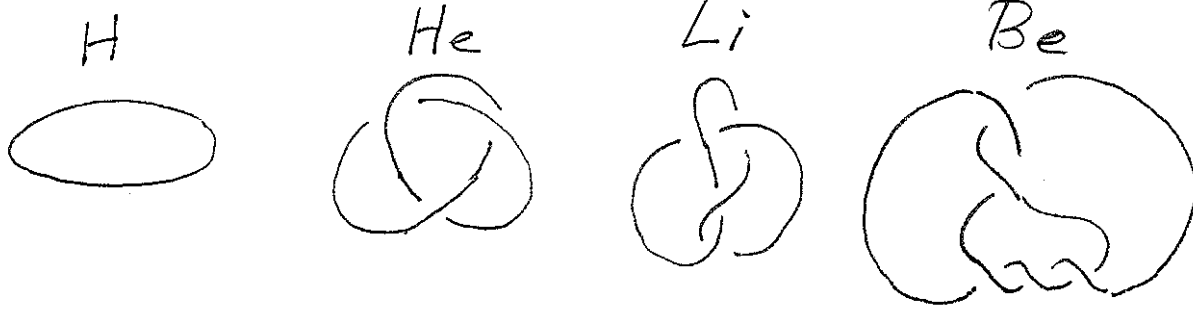


which is actually orientable (normals point at you for both discs).

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A closed curve (w/o self intersections) is called a knot. Part of a branch of mathematics called topology.

History: Lord Kelvin: atoms as knots in the "ether" 121  
Tait: made a table of simple knots (1870s)



Actually, there's no ether. (Michelson-Morley 1880s)  
Einstein 1905.

Mathematicians thought about knots anyway for 100 years. Now used in biology studying action of enzymes on DNA....

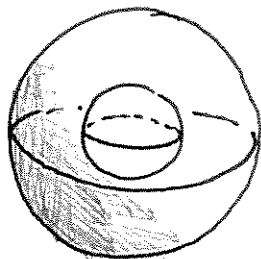
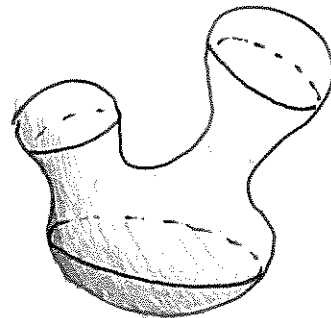
Public key cryptography...

Least area surfaces...

[More generally have...]

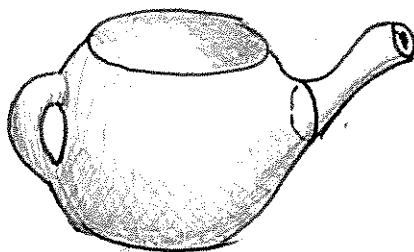
Thm B''. A vector field  $\vec{F}$  on an open simply connected  $R$  in  $\mathbb{R}^3$  is conservative if and only if  $\text{curl } \vec{F} = \vec{0}$  on  $R$ .

Simply connected:



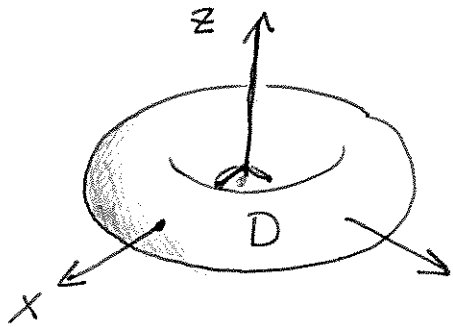
$$1 \leq x^2 + y^2 + z^2 \leq 1$$

Not simply connected:



## Cohomology: Counting holes using vector fields

Consider: The vector field  $\vec{F} = \frac{1}{x^2+y^2}(-y, x, 0)$



makes sense on  $D$  and

has  $\text{curl } \vec{F} = \vec{0}$  but is not

conservative. Turns out

that if  $\vec{G}$  on  $D$  has  $\text{curl } \vec{G} = \vec{0}$  then  $\vec{G} = c\vec{F} + \nabla f$  for some  $c \in \mathbb{R}$  and  $f: D \rightarrow \mathbb{R}$ . So there's

basically one irrotational vector that is not

conservative  $\rightsquigarrow D$  has exactly one

"1-dimensional hole".

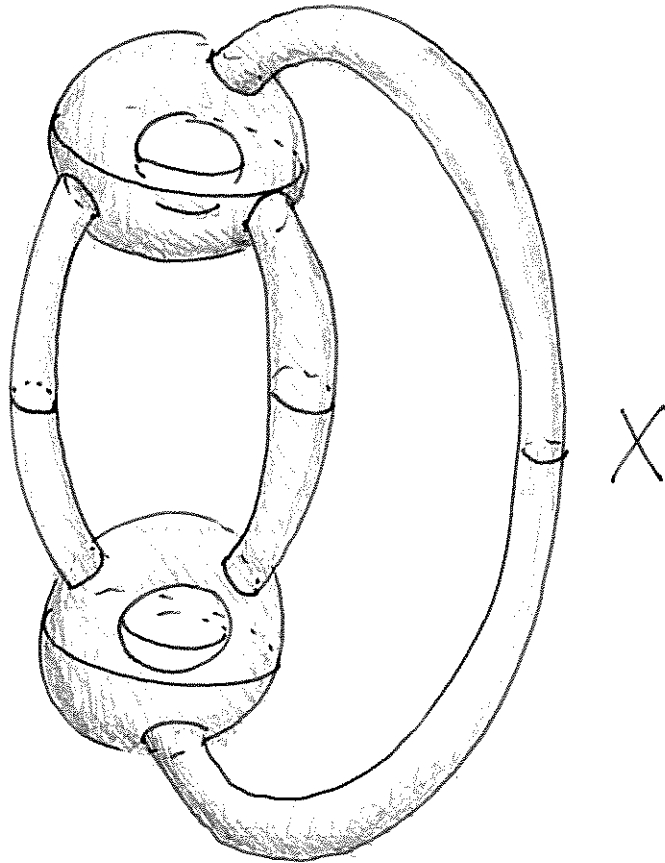
Count "2-dim holes" by looking at incompressible

$\vec{F}$  (i.e.  $\text{div } \vec{F} = 0$ ) that aren't equal to

$\text{curl } \vec{G}$  for any vector field  $\vec{G}$ .

Surprising Key:  $\text{curl}(\nabla f) = \vec{0}$  for any  $f$   
 $\text{div}(\text{curl } \vec{F}) = \vec{0}$  for any  $\vec{F}$ .

Ex:



Has 2 "1-dim holes"  
and 2 "2-dim holes"