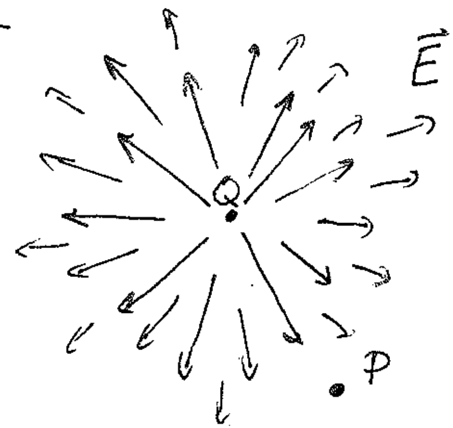
Lecture 39: Electrostatics and Gauss's Law (§16.9)

Previously: Divergence Thm: D a region in $\mathbb{R}^3$, $\vec{F}$ a vector field

$$\iint_{\partial D} (\vec{F}\cdot\vec{n})\, dA = \iiint_D \operatorname{div}\vec{F}\, dV$$

Electrostatics: Particle of charge Q at $\vec{0}$.

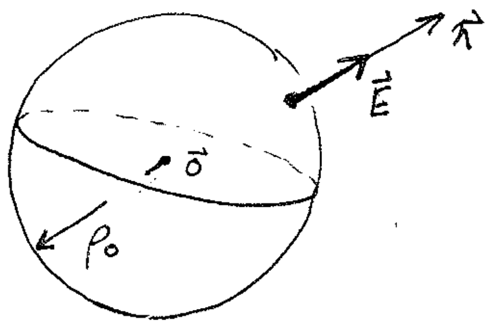


Electric field at $\vec{r} = (x,y,z)$:

$$\vec{E}(\vec{r}) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{|\vec{r}|^3}\vec{r} \quad \left[\begin{array}{c}\text{inverse}\\ \text{square law}\end{array}\right]$$

Force experienced by a charge P at positron $\vec{r}$: $\vec{F} = P E(\vec{r})$.

Flux:



S = Sphere of radius ρ_0

$$\iint_S (E\cdot\vec{n})\, dA = \iint_{S_{\rho_0}} |\vec{E}|\, dA$$

$$= \iint_S \frac{Q}{4\pi\varepsilon_0\rho_0^2}\, dA = \frac{Q}{\varepsilon_0}$$

Note: We calculated Area() $= 4\pi$, so

Area() $= 4\pi\rho_0^2$.

Let D be the region bounded by S, and lets check this by calculating $\iiint_D \operatorname{div}\vec{F}\, dV$

Now $\vec{E} = \dfrac{Q}{4\pi\varepsilon_0}\left(\dfrac{x}{\rho^3}, \dfrac{y}{\rho^3}, \dfrac{z}{\rho^3}\right)$ where $\rho = |\vec{r}| = \sqrt{x^2+y^2+z^2}$

and noting that $\dfrac{\partial \rho}{\partial x} = \frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2x) = \dfrac{x}{\rho}$

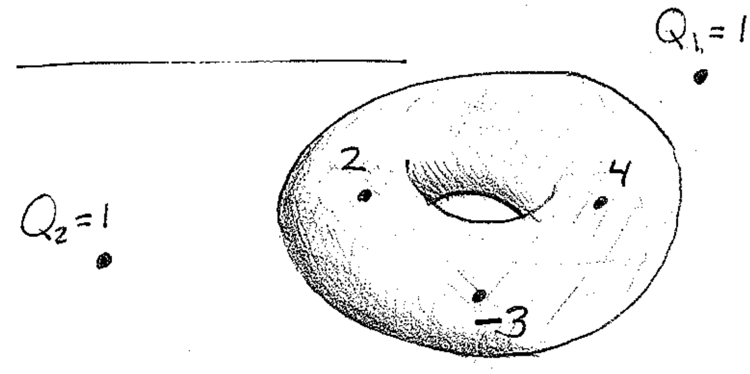
we find

$$\operatorname{div} \vec{E} = \frac{\partial E_1}{\partial x} + \frac{\partial E_2}{\partial y} + \frac{\partial E_3}{\partial z} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{\rho^3 - 3\rho^2 x^2}{\rho^6} + \frac{\rho^3-3\rho^2y^2}{\rho^6} + \frac{\rho^3-3\rho^2z^2}{\rho^6}\right)$$

$$= 0.$$

So it appears that $\iiint_D \operatorname{div} \vec{E}\, dV = 0$, violating the Divergence Thm??? What's gone wrong?

A. $\vec{E}$ isn't defined on all of D [since it doesn't make sense at $\vec{0}$.]



Suppose have charges Q_i at positions $\vec{p}_i$

$$\vec{E}(\vec{r}) = \sum E_i(\vec{r}) = \sum \frac{Q_i}{4\pi\varepsilon_0}\frac{1}{|\vec{r}-\vec{p}_i|^3}(\vec{r}-\vec{p}_i)$$

Gauss's Law: Let D be a region in $\mathbb{R}^3$. Then

$$\iint_{\partial D} (\vec{E}\cdot\vec{n})\, dA = \frac{1}{\varepsilon_0}\left(\sum_{\substack{i \text{ where } \vec{p}_i \\ \text{is in } D}} Q_i\right)$$

Ex: In the above pic, $\iint_{\partial D} (\vec{F} \cdot \vec{n})\, dA = \frac{1}{\varepsilon_0}(2+4-3) = \frac{3}{\varepsilon_0}$.

Reason [for Gauss's Law:] First notice

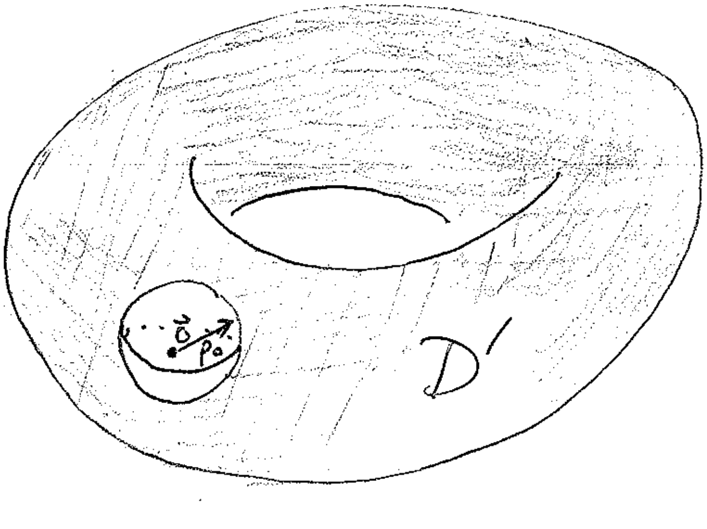
$$\iint_{\partial D} (\vec{E} \cdot \vec{n})\, dA = \iint_{\partial D} \left(\sum E_i\right) \cdot \vec{n}\, dA = \sum \iint_{\partial D} (E_i \cdot \vec{n})\, dA$$

So assume we have a single charge Q at $\vec{o}$. (after changing coordinates), and hence $\vec{E} = \frac{Q}{4\pi \varepsilon_0} \frac{1}{\rho^3} \vec{r}$

Key: $\iint_{\partial D} (\vec{E} \cdot \vec{n})\, dA = \begin{cases} 0 & \text{if } \vec{o} \text{ is not in } D. \\ Q/\varepsilon_0 & \text{if } \vec{o} \text{ is in } D. \end{cases}$

Point: If $\vec{o}$ is not in D, then $\operatorname{div} \vec{E} = 0$ on all of D, and hence $\iint_{\partial D} (\vec{E} \cdot \vec{n})\, dA = \iiint_D \operatorname{div} \vec{E}\, dV = 0$.

If $\vec{o}$ is in D, consider $D' = D$ with ball of radius ρ_0 about $\vec{o}$ removed.



Then

$$0 = \iiint_{D'} \operatorname{div} \vec{E}\, dV =$$

$$\iint_{\partial D'} (\vec{E} \cdot \vec{n})\, dA =$$

inward normal

$$\iint_{\partial D} \vec{E}\cdot\vec{n}\, dA + \iint_{S_{\rho_0}} \vec{E}\cdot\vec{n}\, dA$$

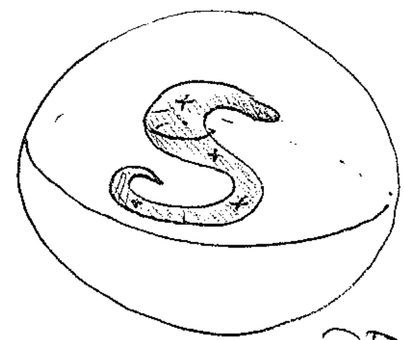
outward normal

So

outward normal.

$$\iint_{\partial D} \vec{E}\cdot\vec{n}\, dA = \iint_{S_{\rho_0}} \vec{E}\cdot\vec{n}\, dA = \frac{Q}{\varepsilon_0}.$$

While charge *is* quantized, when there are many charges, look at the charge density: $\rho(x,y,z)$ (units: $\frac{\text{coulomb}}{m^3}$). By Gauss's Law

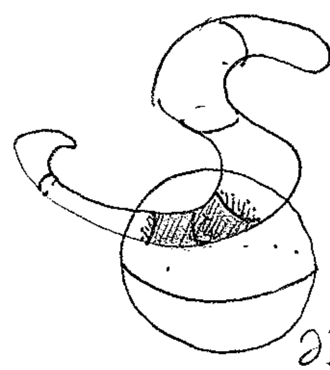


∂D

$$\iint_{\partial D} (\vec{E}\cdot\vec{n})\, dA = \frac{1}{\varepsilon_0} \iiint_D \rho\, dV$$

$$\parallel$$

$$\iiint_D \operatorname{div} \vec{E}\, dV$$



∂D

Since this is true for *all* D, must have

$$\operatorname{div} \vec{E} = \frac{1}{\varepsilon_0}\rho.$$