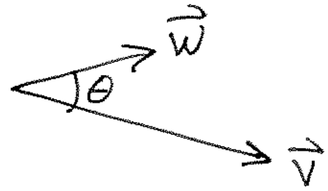
# Lecture 4: Cross Product (§12.4)

**Previously...** $\vec{v} = (v_1, v_2, v_3)$ $\vec{w} = (w_1, w_2, w_3)$

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3 = |\vec{v}||\vec{w}|\cos\theta$$

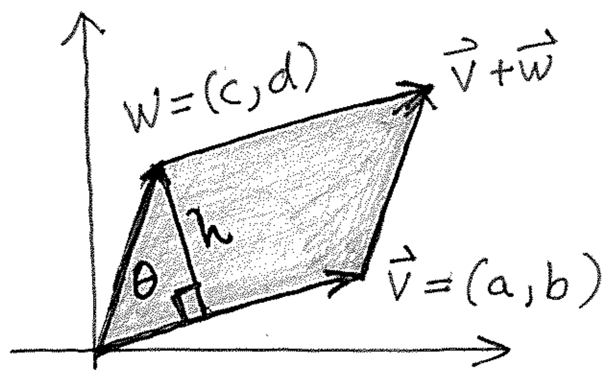



---

[Today, will talk about a second way to multiply vectors and give some applications, including a prob. about planes.]

**Determinant:** $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Gives area of:

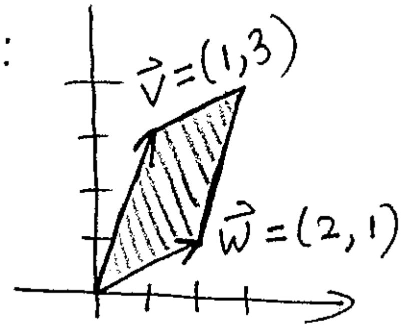



$$A = (\text{base})(\text{height}) = |\vec{v}||\vec{w}|\sin\theta$$

**Reason:**

$$A^2 = |\vec{v}|^2|\vec{w}|^2\sin^2\theta = |\vec{v}|^2|\vec{w}|^2(1 - \cos^2\theta)$$
$$= |\vec{v}|^2|\vec{w}|^2 - (\vec{v} \cdot \vec{w})^2$$
$$= (a^2 + b^2)(c^2 + d^2) - (ac + bd)^2$$
$$= a^2d^2 + b^2c^2 - 2abcd = (ad - bc)^2 = \begin{vmatrix} a & b \\ c & d \end{vmatrix}^2$$

Ex:




$$\text{Area} = \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 1-2\cdot 3 = -5$$

Negative area?!

Switch $\vec{v}$ and $\vec{w}$: $\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 6-1 = 5$

In general

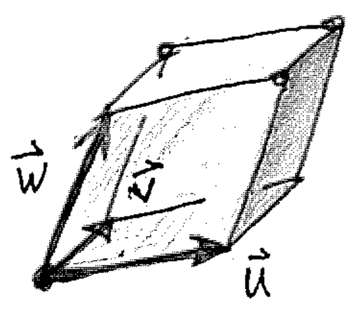
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{cases} \text{area} & \text{if } \vec{w} \text{ is anticlockwise from } \vec{v} \\ -\text{area} & \text{if } \vec{w} \text{ is clockwise from } \vec{v} \end{cases}$$

[First example of the right-hand rule... ]

---

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

= Signed volume of this parallelepiped in $\mathbb{R}^3$.




[Will come back to the sign and why this works.]

**Cross Product:** $\vec{v} = (v_1, v_2, v_3) = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$

$\vec{w} = (w_1, w_2, w_3)$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}\vec{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}\vec{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}\vec{k}$$

**Ex:** $\vec{v} = (0,1,3)$ $\vec{w} = (2,1,1)$

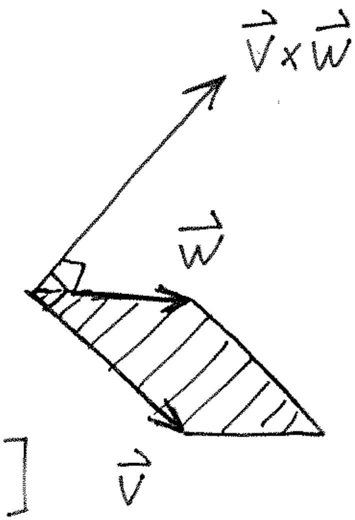
$$\vec{v}\times\vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 3 \\ 2 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix}\vec{i} - \begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix}\vec{j} + \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix}\vec{k}$$

$$= -2\vec{i} + 6\vec{j} - 2\vec{k} = (-2, 6, -2)$$

[*Uses*: Maxwell's equations, torque, geom. probs.]

**Properties:**

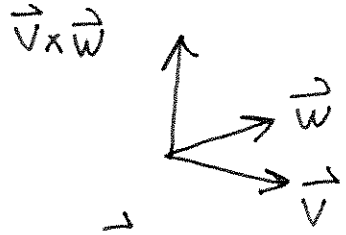
1) $\vec{v}\times\vec{w}$ is orthogonal to $\vec{v}$ and $\vec{w}$.
2) $|\vec{v}\times\vec{w}| = |\vec{v}||\vec{w}|\sin\theta$
   = Area of parallelogram:

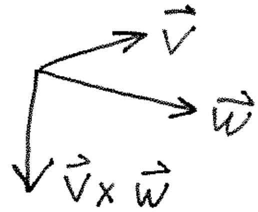



3) $\vec{v}\times\vec{w} = -\vec{w}\times\vec{v}$ [Not commutative!]
4) $\vec{v}\times\vec{v} = 0$
5) $\vec{u}\times(\vec{v}+\vec{w}) = \vec{u}\times\vec{v} + \vec{u}\times\vec{w}$

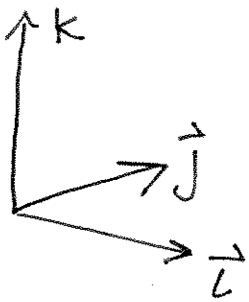
**Note:** ① and ② almost determine $\vec{v}\times\vec{w}$.

The rest comes from the right-hand rule:




**Ex:**




$\vec{i}\times\vec{j}=\vec{k}$ $\qquad$ $\vec{j}\times\vec{k}=\vec{i}$

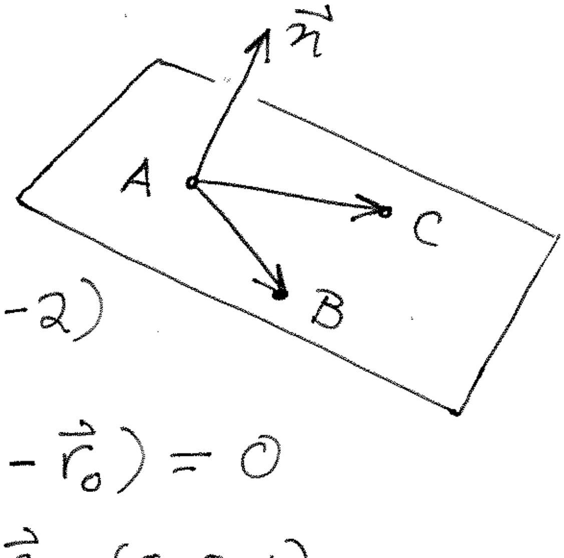
$\vec{i}\times\vec{k}=-\vec{j}$

**Not Associative:** $\vec{i}\times(\vec{i}\times\vec{j})=\vec{i}\times\vec{k}=-\vec{j}$

$$(\vec{i}\times\vec{i})\times\vec{j}=\vec{0}\times\vec{j}=\vec{0}$$

---

**Ex:** Find the equation of the plane containing $A=(0,0,1)$, $B=(0,1,4)$, and $C=(2,1,2)$




$\vec{v}=\overrightarrow{AB}=(0,1,3)$

$\vec{w}=\overrightarrow{AC}=(2,1,1)$

Normal: $\vec{n}=\vec{v}\times\vec{w}=(-2,6,-2)$

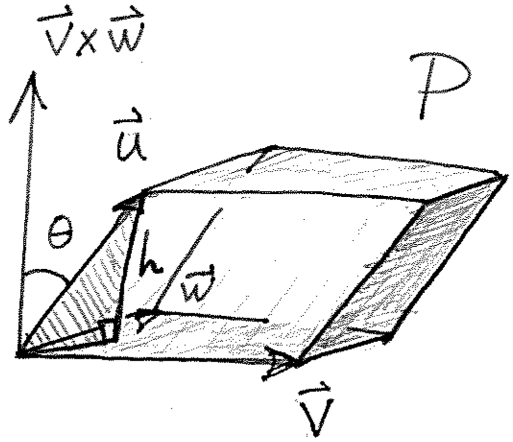
Equation for plane: $\vec{n}\cdot(\vec{r}-\vec{r}_0)=0$

where $\vec{r}=(x,y,z)$ and $\vec{r}_0=(0,0,1)$.

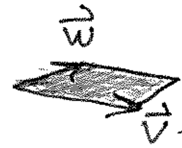
Expanding out gives: $-2x+6y-2z+2=0$.

---

<u>Triple Product</u>: $\vec{u}\cdot(\vec{v}\times\vec{w})$




$$\vec{u}\cdot(\vec{v}\times\vec{w}) = |\vec{u}|\,|\vec{v}\times\vec{w}|\cos\theta$$

$= (|\vec{u}|\cos\theta)$ (Area of )

$= \begin{pmatrix}\text{height}\\ \text{of } P\end{pmatrix}\begin{pmatrix}\text{Area of base}\\ \text{of } P\end{pmatrix} = \text{Volume of } P$

$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

[To check, expand out $\vec{u}\cdot(\vec{v}\times\vec{w})$ and the determinant.]

---

Reasons for the properties of the cross product (<u>only if asked</u>).

1. Compute $\vec{v}\cdot(\vec{v}\times\vec{w})$ and $\vec{w}\cdot(\vec{v}\times\vec{w})$ and find you get 0.
2. Similar to $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \text{Area}$, see text for details.

(3-5) Direct calculations.

Q: Is there a better way to multiply vectors in $\mathbb{R}^3$? Would like it to be associative and have $\vec{v} \star \vec{w} \neq \vec{0}$ if $\vec{v}, \vec{w} \neq \vec{0}$.

A: No. All there is: $\mathbb{R} = \mathbb{R}^1$ $\quad \mathbb{C} = \mathbb{R}^2$

$\mathbb{H} = \mathbb{R}^4$ $\quad \mathbb{O} = \mathbb{R}^8$

Connected to the fact that no matter how the wind blows, there is always somewhere on Earth where it is completely still.