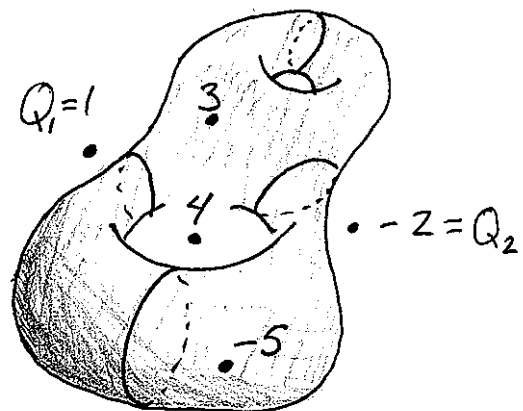


Lecture 40: Maxwell's equations

①

Last time: Charges Q_i at positions \vec{p}_i

$$\vec{E}(\vec{r}) = \sum \frac{Q_i}{4\pi\epsilon_0} \frac{1}{|\vec{r} - \vec{p}_i|^3} (\vec{r} - \vec{p}_i)$$



Gauss's Law: D a region in \mathbb{R}^3

$$\iint_{\partial D} (\vec{E} \cdot \vec{n}) dA = \frac{1}{\epsilon_0} (\text{Total charge in } D)$$

$$\text{Flux} = \frac{-2}{\epsilon_0}$$

On HW for today, used this compute the total charge from a formula for \vec{E} . This is not impractical...

When there are many (e.g. 10^{20}) can't use ~~the~~ the formula for \vec{E} directly. Don't want to focus on each charge individually just as we don't count molecules when measuring the mass.

Mass: $\rho(x, y, z)$ mass density, units = $\frac{g}{m^3}$



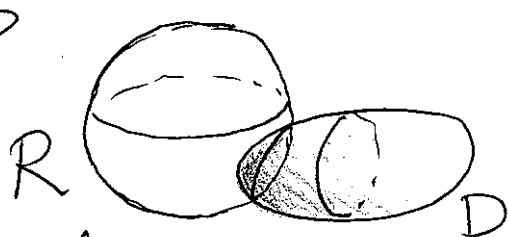
D

$$\text{Total Mass of } D = \iiint_D \rho dV$$

Charge: $\rho(x, y, z)$ charge density, units = $\frac{\text{Coulomb}}{m^3}$

$$\text{Total Charge of } D = \iiint_D \rho dV$$

Q: How does ρ determine \vec{E} ?



Gauss's Law should still hold, so for a region R we have

$$\frac{1}{\epsilon_0} \left(\begin{array}{c} \text{Charge} \\ \text{in } R \end{array} \right) = \iint_{\partial R} (\vec{E} \cdot \vec{n}) dA = \iiint_R \text{div } \vec{E} dV$$

" ↑ Divergence Thm

$$\frac{1}{\epsilon_0} \iiint_R \rho dV$$

As true for all regions R , must have $\text{div } \vec{E} = \frac{\rho}{\epsilon_0}$

[Q: Does this answer the question? Not completely since]
many vector fields have the same divergence.

A. $\vec{E}(\vec{r}) = (E_1(\vec{r}), E_2(\vec{r}), E_3(\vec{r}))$ where if $\vec{r} = (a, b, c)$

then

$$E_1(\vec{r}) = \frac{1}{4\pi\epsilon_0} \iiint_D \frac{(a-x)\rho(x,y,z)}{|\vec{r} - (x,y,z)|^3} dV$$

etc.

Exercise: Take $D = \text{unit sphere}$ and $\rho = 1$.

Use above to calculate the electric field at $\vec{r} = (a, b, c)$

Hint: Use symmetry to reduce to the case where

$$\vec{r} = (a, 0, 0).$$

Maxwell's Equations:

(3)

$\vec{E}(x, y, z, t)$ - Electric field (at time t) ($\mathbb{R}^4 \rightarrow \mathbb{R}^3$)

$\vec{B}(x, y, z, t)$ - Magnetic field

$\rho(x, y, z, t)$ - charge density ($\mathbb{R}^4 \rightarrow \mathbb{R}$)

Gauss's Law:

$$\text{div } \vec{E} = \frac{\rho}{\epsilon_0}$$

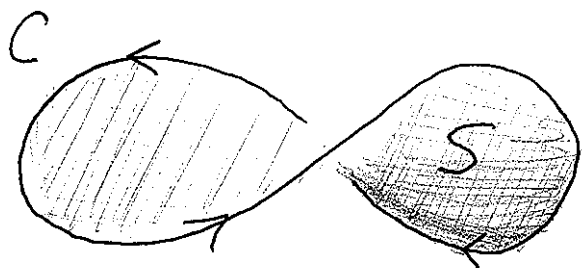
$$\iint_{\partial R} \vec{E} \cdot \vec{n} \, dA = \iiint_R \rho \, dV$$

Gauss's Law for magnetic fields: [No magnetic monopoles.]

$$\text{div } \vec{B} = 0$$

$$\iint_{\partial R} (\vec{B} \cdot \vec{n}) \, dA = 0$$

Faraday's Law of Induction: A changing magnetic field induces a current in a loop of wire.



Now

$$\underbrace{\int_C \vec{E} \cdot d\vec{r}}_{\text{electromotive force aka voltage}} = -\frac{\partial}{\partial t} \iint_S \vec{B} \cdot \vec{n} \, dA$$

$$\int_C \vec{E} \cdot d\vec{s} = \iint_S (\text{curl } \vec{E}) \cdot \vec{n} \, dA \quad \text{and so}$$

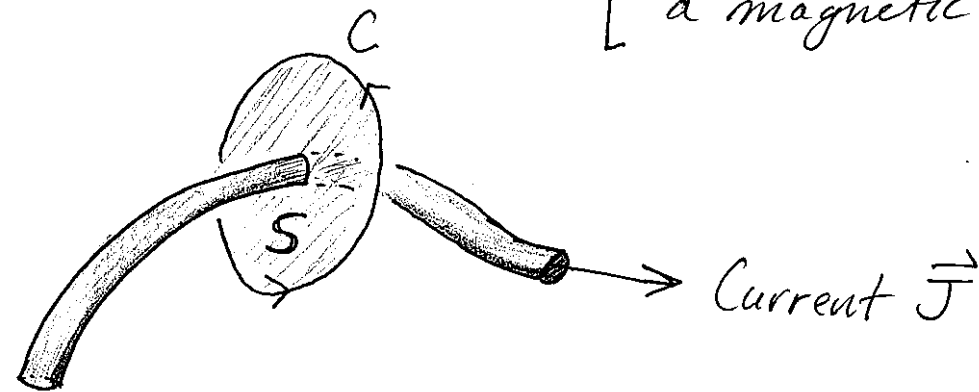
$$\boxed{\text{curl } \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

Unit check: (a) $\vec{F} = Q\vec{E} \Rightarrow \vec{E} \text{ in } \frac{N}{C} = \frac{V}{m} \Rightarrow \int_C \vec{E} \cdot \frac{d\vec{r}}{m} \text{ is in volts.}$ (4)

(b) \vec{B} has units $T = \text{Tesla} = \frac{Vs}{m^2}$ $\vec{n} = \text{unitless}$

So $\iint_S \underbrace{\vec{B} \cdot \vec{n}}_{m^2} dA$ is in $Vs \Rightarrow \frac{\partial}{\partial t} \iint_S (\vec{B} \cdot \vec{n}) dA$ is in V .

Ampere's circuital law: [Current in a wire or a changing electric field induces a magnetic field.]



$$\int_C \vec{B} \cdot d\vec{r} = \mu_0 \iint_S (\vec{J} \cdot \vec{n}) dA + \epsilon_0 \mu_0 \frac{\partial}{\partial t} \iint_S (\vec{E} \cdot \vec{n}) dA$$

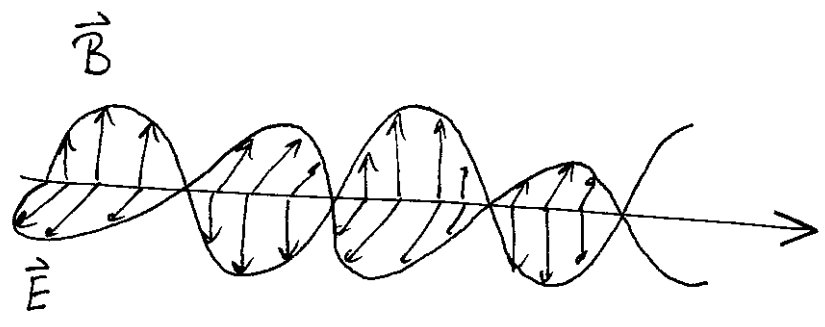
$$\iint_S (\text{curl } \vec{B}) \cdot \vec{n} dA$$

So $\boxed{\text{curl } \vec{B} = \mu_0 \vec{J} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}}$

$\epsilon_0 = \text{permittivity of free space}$
 $= \text{farads/m}$

$\mu_0 = \text{permeability of free space} = \text{N/A}^2$

And there was light...



$c = \text{speed of light}$
 $= \frac{1}{\sqrt{\mu_0 \epsilon_0}}$

$c = \omega \lambda$ ← wave length
 ↑ freq

$\vec{E} = (0, c \cdot \cos(2\pi(\omega t - x/\lambda)), 0)$

$\vec{B} = (0, 0, \cos(2\pi(\omega t - x/\lambda)))$

Check: $\text{curl } \vec{E} = \frac{\partial \vec{B}}{\partial t}$ $\text{curl } \vec{B} = \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$
 $\text{div } \vec{E} = 0$ $\text{div } \vec{B} = 0.$

Moral: Here are other applications of line and surface integrals, as well as the relations between them.

If time remains: (a) Multitude of integral theorems.

(b) $\int_{\partial M} \omega = \int_M d\omega$