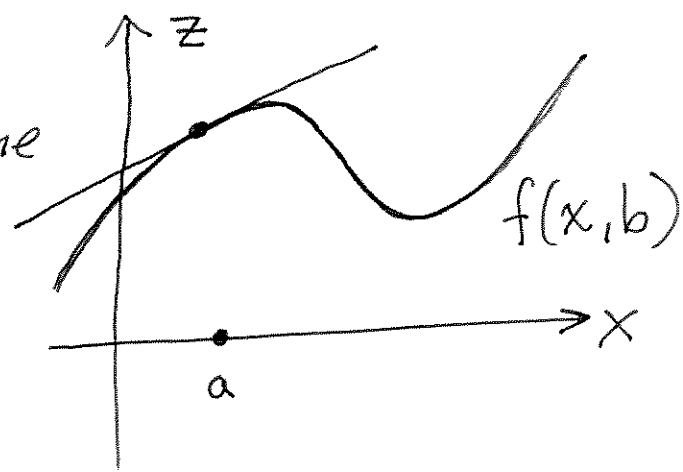
Lecture 9: Partial Derivatives and Applications (§14.3-4)

Last time: $f: \mathbb{R}^2 \to \mathbb{R}$

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

= rate of change as move in x-direction from (a,b)

= slope of tangent line in slice.



Compute by taking other vars as constant

$$\frac{\partial}{\partial x}(x^3 y + xy) = 3x^2 y + y$$

Higher order: $$\frac{\partial}{\partial y \partial x}(x^3 y + xy) = \frac{\partial}{\partial y}(3x^2 y + y) = 3x^2 + 1$$

[All partial derivatives together will play the same role as the derivatives for one-var calc.

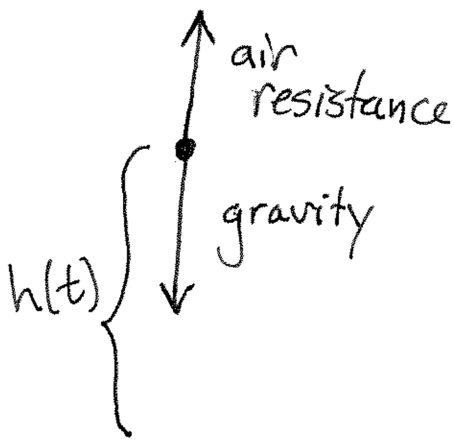
(e.g. min/max, tangent planes, Taylor series]

O.D.E. Ordinary Differential Equations.

1. $p(t)$ = population at time t

$$\frac{dp}{dt}(t) = c\,p(t) \implies p(t) = p_0 e^{ct}$$

2. [Skip!] $h''(t) = -g - a h'(t)$

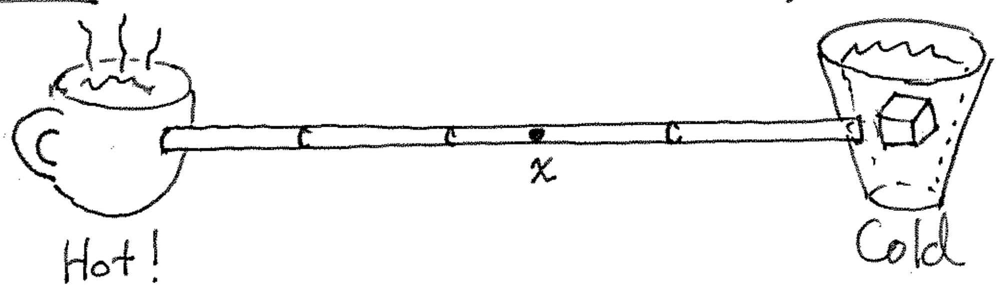


$$h(t) = -\frac{1}{a^2}\left(agt + (av_0 + g)e^{-at}\right)$$

Terminal velocity: $-g/a$

[Subject of Math 285/286.]

P.D.E. Partial Differential Equations.

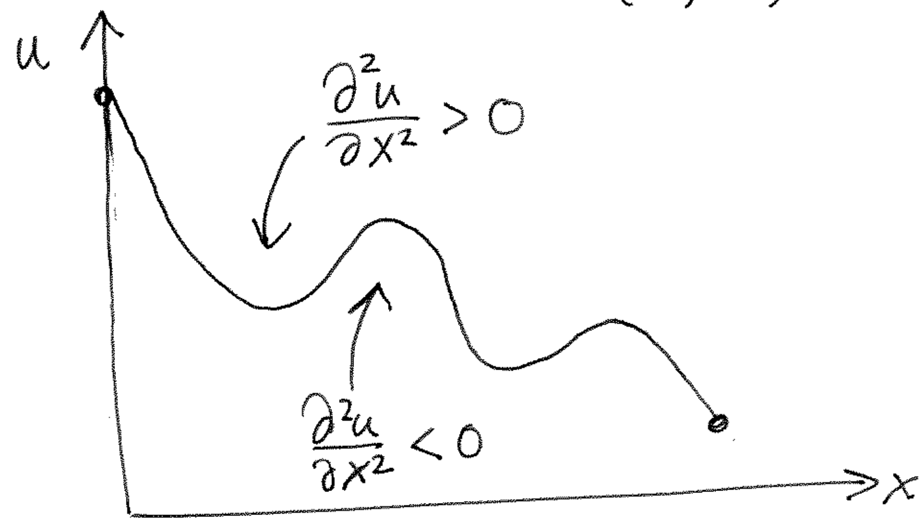


$u(x,t)$ = temperature of rod at pos x and time t.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) \quad \text{[Heat equation]}$$

Comes from Newton's Law of Cooling: Heat flow is proportional to $-\frac{\partial u}{\partial x}$

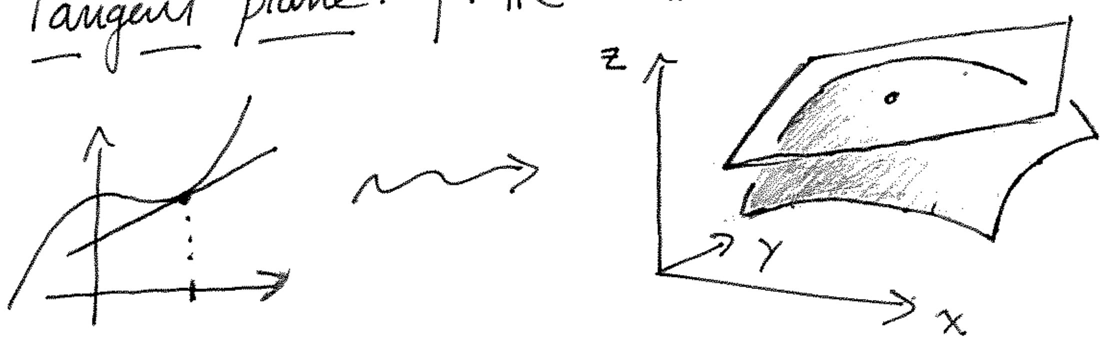
Slice at time t_0: $\quad u(x,t_0) = V(x)$



$$\frac{\partial^2 u}{\partial x^2}(x,t_0) = V''(x)$$

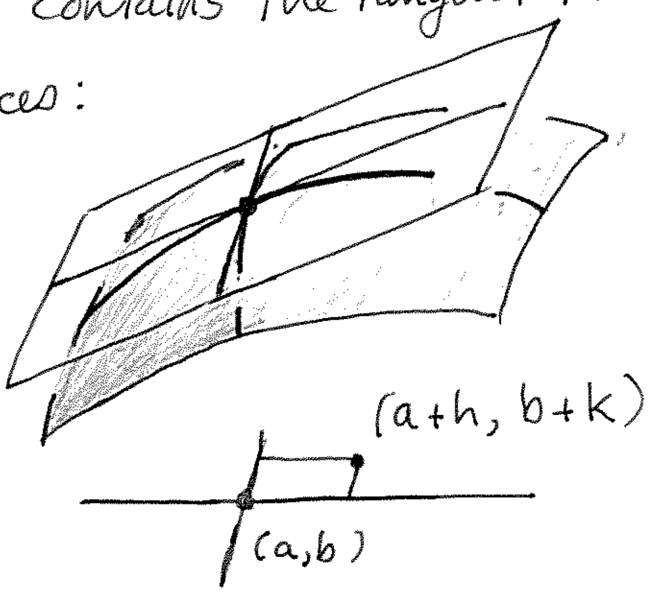
[Over time, the heat distribution tends to spread out.]
Show visualization.

Tangent plane: $f : \mathbb{R}^2 \to \mathbb{R}$



The tangent plane contains the tangent lines in the x and y slices:

To find the formula, approx. f as follows



$$f(a+h,\, b+k) = f(a,b) + \frac{\partial f}{\partial x}(a,b)\,h + \frac{\partial f}{\partial y}(a,b)\,k + E(h,k)$$

Say that f is <u>differentiable</u> at (a,b) if

$$\lim_{(h,k)\to\vec{0}} \frac{|E(h,k)|}{\sqrt{h^2+k^2}} = 0.$$ Means that there is a tangent plane at $(a,b,f(a,b))$. Rewrite

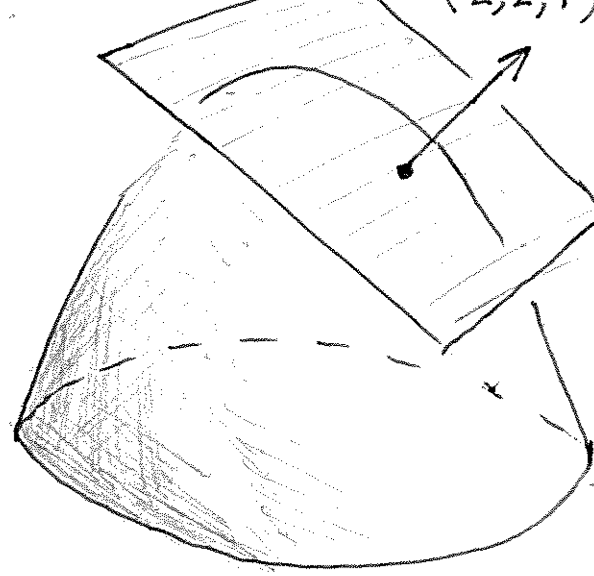
$$f(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b) + \text{Error}$$

So the tangent plane is given by

$$z - f(a,b) = \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

<u>Ex</u>: $f(x,y) = -x^2 - y^2$

$$\frac{\partial f}{\partial x} = -2x \quad \frac{\partial f}{\partial y} = -2y$$



Tangent plane at $(1,1,-2)$

$$z + 2 = -2(x-1) - 2(y-1)$$

$$\iff 2x + 2y + z = 2$$

Warning: Just because $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at (a,b) does not mean f is differentiable at (a,b).

Ex: $f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

$\frac{\partial f}{\partial y}(0,0)$ is also 0.

But: f isn't continuous at $(0,0)$ as $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. So f can't be differentiable by

Thm: If $f: \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (a,b) then f is continuous at (a,b).

Reason: As f is diff at (a,b) have

$$f(a+h, b+k) = f(a,b) + \frac{\partial f}{\partial x}(a,b)h + \frac{\partial f}{\partial y}(a,b)k + E(h,k)$$

where $\lim_{(h,k)\to(0,0)} \frac{E(h,k)}{\sqrt{h^2+k^2}} = 0$. Thus as $(h,k) \to (0,0)$

we have $f(a+h, b+k) \to f(a,b) + 0 + 0 + 0$.

So $\lim_{(x,y)\to(0,0)} f(x,y) = f(a,b)$ and f is continuous at (a,b).

<u>Thm</u>: $f: \mathbb{R}^2 \to \mathbb{R}$. Suppose $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist near (a,b) and are continuous there. Then f is differentiable at (a,b).

<u>Ex</u>: $f(x,y) = xy^3 + xy + \sin(xy)$ is differentiable on all of $\mathbb{R}^2$.