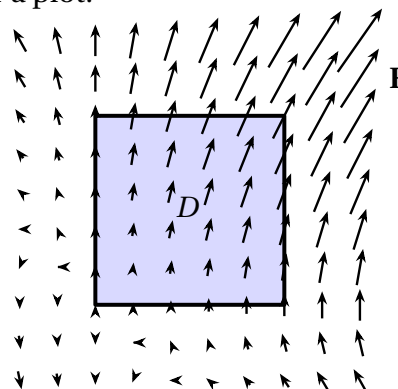


Thursday, November 14 ** Green's Theorem

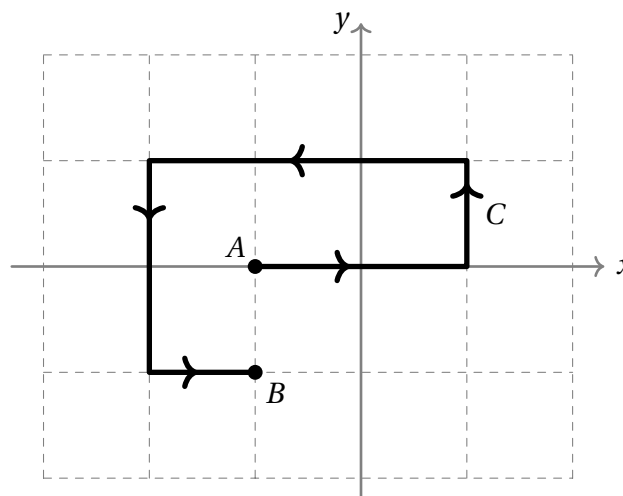
Green's Theorem is a 2-dimensional version of the Fundamental Theorem of Calculus: it relates the (integral of) a vector field \mathbf{F} on the boundary of a region D to the integral of a suitable *derivative* of \mathbf{F} over the whole of D .

- Let D be the unit square with vertices $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$ and consider the vector field $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y) \rangle = \langle xy, x+y \rangle$. See below right for a plot.

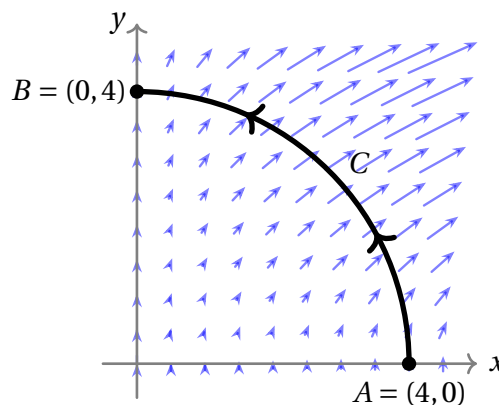
- For the curve $C = \partial D$ oriented counter-clockwise, directly evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. Hint: to speed things up, have each group member focus on one side of C .
- Now compute $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$.
- Check that Green's Theorem works in this example.



- Compute the line integral of $\mathbf{F}(x,y) = \langle x^3, 4x \rangle$ along the path C shown at right against a grid of unit-sized squares. To save work, use Green's Theorem to relate this to a line integral over the vertical path joining B to A . Hint: Look at the region D bounded by these two paths. Check your answer with the instructor.

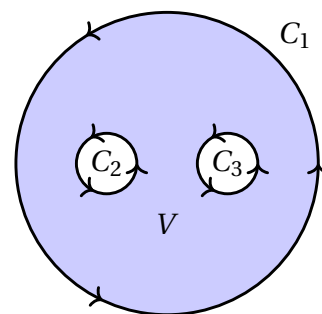


- Consider the quarter circle C shown below and the vector field $\mathbf{F}(x,y) = \langle 2xe^y, x+x^2e^y \rangle$. The goal of this problem is to compute the line integral $I_0 = \int_C \mathbf{F} \cdot d\mathbf{r}$.



- Parameterize C and start directly expanding out I_0 into an ordinary integral in t until you are convinced that finding I_0 this way will be a highly unpleasant experience.
- Check that \mathbf{F} is *not* conservative, so we can't use that trick directly to compute I_0 .
- Find a function $f(x, y)$ such that $\mathbf{F} = \mathbf{G} + \nabla f$, where \mathbf{G} is the vector field $\langle 0, x \rangle$.
- Argue geometrically that \mathbf{G} integrates to 0 along any line segment contained in either the x -axis or the y -axis.
- Use part (d) with Green's Theorem to show that $\int_C \mathbf{G} \cdot d\mathbf{r} = 4\pi$.
- Combine parts (c–e) with the Fundamental Theorem of Line Integrals to evaluate I_0 . Check your answer with the instructor.

4. Consider the shaded region V shown, bounded by a circle C_1 of radius 5 and two smaller circles C_2 and C_3 of radius 1. Suppose $\mathbf{F}(x, y) = \langle P, Q \rangle$ is a vector field where $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2$ on V . Assuming in addition that $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 3\pi$ and $\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = 4\pi$, compute $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$. Check your answer with the instructor.



5. Suppose D is a region in the plane bounded by a closed curve C . Use Green's Theorem to show that both $\int_C x \, dy$ and $-\int_C y \, dx$ are equal to $\text{Area}(D)$.
6. The curve satisfying $x^3 + y^3 = 3xy$ is called the *Folium of Descartes* and is shown at right.

- Let C be the “bulb” part of this folium, more precisely, the part in the positive quadrant. Show that any line $y = tx$ for $t > 0$ meets C in exactly two points, one of which is the origin. Use this fact to parameterize C by taking the slope t as the parameter.
- Use part (a) and Problem 5 to compute the area bounded by C . Check your answer with the instructor.

