Lecture 35:

**\( \Omega \)-spectrum:** A sequence of CW complexes \( \{K_n\} \) together with homotopy equivalences \( K_n \to \Omega K_{n+1} \).

**Thm:** If \( K_n \) is an \( \Omega \)-spectrum, then \( h^n(X) = \langle X, K_n \rangle \) defines a reduced cohomology theory of based CW complexes.

Brown Representability: Any reduced cohomology theory of CW complexes comes from some \( \Omega \)-spectrum.

**Thm.** G abelian group. There is a class \( \alpha_n \in H^n(K(G,n); G) \) so that

\[
T: \langle X, K(G,n) \rangle \to H^n(X; K)
\]

\[
f \mapsto f^*(\alpha_n)
\]

is an isomorphism for all CW complexes \( X \).

**Pf.** Let \( K_n \) be the Eilenberg-MacLane \( \Omega \)-spectrum. By theorem, this gives a cohomology theory \( h^* \).

**Now** \( h^*(S^n) = \begin{cases} 0 & * \neq 0 \\ \langle S^n, K_0 \rangle \cong G & * = 0 \end{cases} \)

\( S_0 \) set of pts of size \( G \)

where the group structure on \( \langle S^n, K_0 \rangle \) comes from
the homotopy equivalence $K_0 \cong \Omega \cdot K_1$. By uniqueness of cohomology for CW complexes with this value for $h^*(S^0)$, have $h^*(X) \cong H^*(X; G)$ via a natural isomorphism $T$.

Define $\alpha_n$ as the image of $\text{id}_{K(n,n)}$ under $T$. Then for $f \in \langle X, K(n,n) \rangle$ we have

$$T([f]) = T([\text{id}_{K(n,n)} \circ f]) = T(f^*[\text{id}_{K(n,n)}])$$

$$\quad = f^*T([\text{id}_{K(n,n)}]) = f^*(\alpha_n).$$

Geometric construction: $K_n$ a $K(G, n)$ with $K_0 = \text{pt}$. Then $\alpha$ is the cellular cochain assigning to an $n$-cell $e$ the corresponding element of $T_n K_n = G$.

[Query: Why is $S\alpha = 0$?]

\[\text{Diagram:}\]
Cup product: \( X \xrightarrow{f} K_n, X \xrightarrow{g} K_m \) where \( G = R \) is a ring.

Q: What is \( X \xrightarrow{f \cup g} K_{n+m} \)?

\[
K_n \wedge K_m = \text{smash product} = K_n \times K_m \xrightarrow{\text{wedge at base points}} \frac{K_n \vee K_m}{\text{base pt of } K_n \wedge K_m}.
\]

Ex: \( S^1 \wedge S^1 = S^2 \)

\[
S^n \wedge S^m = S^{n+m}
\]

Using \( K_k \) with \( K_k^{(k-1)} \) = \( k \) pt, see \( K_n \wedge K_m \) is \( n+m-1 \) conn.

Claim: \( \tilde{H}_{n+m}(K_n \wedge K_m) = R \otimes \mathbb{Z} R \).

Pf. Same as \( \tilde{H}_{n+m}(K_n \wedge K_m; \mathbb{Z}) \) by Hurewicz. By full Künneth theorem for smash products (Hatcher pg 276), get

\[
\tilde{H}_{n+m}(K_n \wedge K_m) = \bigoplus_{i=0}^{n+m} \tilde{H}_i(K_n) \otimes \tilde{H}_{n+m-i}(K_m)
\]

\[
= \tilde{H}_n(K_n) \otimes_{\mathbb{Z}} \tilde{H}_m(K_m) \text{ as the Tor term is } 0.
\]

[Point: With reduced (co)homology, \( X \wedge Y \) is more natural than \( X \times Y \).]
Define $K_n \wedge K_m \overset{m}{\longrightarrow} K_{n+m}$ so that $u_*$ on $T^n \wedge T^m$ is ring mult $R \otimes_\mathbb{Z} R \rightarrow R$.

[On HW will show $\phi: G \rightarrow H$ induces $K(G, n) \rightarrow K(H, n)$, this is the same idea.]

If $f: X \rightarrow K_n$ and $g: Y \rightarrow K_m$, then the cross prod $[f] \times [g]$ in $H^{n+m}(X \times Y)$ is the composition

$$X \times Y \xrightarrow{f \times g} K_n \times K_m \xrightarrow{\mu} K_{n+m}$$

If $g: X \rightarrow K_m$, then $[f] \cup [g]$ is the composition

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} K_n \wedge K_m \overset{\mu}{\longrightarrow} K_{n+m}.$$  

diagonal

Can check the basic props of the cup prod this way. Pesky sign comes down to $S^m \wedge S^n \rightarrow S^n \wedge S^m$ has degree $(-1)^{n \cdot m}$.

That this is really the cup product follows from confirming this for the $a_n$ in $H^n(K_n; G)$.  
