Lecture 35:

<u>Ω-spectrum</u>: A sequence of CW complexes $\{K_n\}$ together with homotopy equivalences $K_n \to \Omega K_{n+1}$.

<u>Thm</u>: If K_n is an Ω-spectrum, then $h^n(X) = \langle X, K_n \rangle$ defines a reduced cohomology theory of based CW complexes.

<u>Brown Representability</u>: Any reduced cohomology theory of CW complexes comes from same Ω-spectrum.

<u>Thm.</u> G abelian group. There is a class $\alpha_n \in H^n(K(G,n); G)$ so that

$$T : \langle X, K(G,n) \rangle \longrightarrow H^n(X; K)$$
$$f \longmapsto f^*(\alpha_n)$$

is an isomorphism for all CW complexes X.

<u>Pf.</u> Let K_n be the Eilenberg-MacLane Ω-spectrum. By theorem, this gives a cohomology theory h^n.

Now $h^*(S^0) = \begin{cases} 0 & * \neq 0 \\ \langle S^0; K_0 \rangle \cong G & * = 0 \end{cases}$

$\underset{s_0}{\bullet} \quad \bullet$ discrete set of pts of size G

where the group structure on $\langle S^0; K_0 \rangle$ comes from the homotopy equivalence $K_0 \simeq \Omega K_1$. By uniqueness of cohomology for CW complexes with this value for $h^*(S^0)$, have $h^*(X) \cong H^*(X; G)$ via a <u>natural</u> isomorphism T.

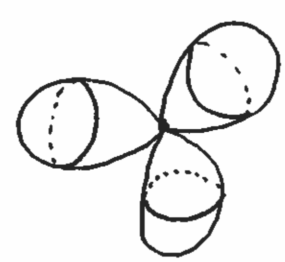
Define α_n as the image of $id_{K(G,n)}$ under T. Then for $f \in \langle X, K(G,n) \rangle$ we have

$$T([f]) = T([id_{K(G,n)} \circ f]) = T(f^*([id_{K(G,n)}]))$$
$$= f^* T([id_{K(G,n)}]) = f^*(\alpha_n).$$ ▨

<u>Geometric construction:</u> K_n a $K(G,n)$ with $K_n^{(n-1)} = pt$.

Then α is the cellular cochain assigning to an n-cell e_α the corresp. elt of $\pi_n K_n = G$.

[Query: Why is $\delta\alpha = 0$?]



<u>Cup product</u>: $X \xrightarrow[f]{} K_n$, $X \xrightarrow[g]{} K_m$ where $G = R$ a ring.

<u>Q</u>: What is $X \xrightarrow[f\cup g]{} K_{n+m}$?

$K_n \wedge K_m =$ smash product $= K_n \times K_m \big/ \underbrace{K_n \vee K_m}_{\text{base pt of } K_n \wedge K_m}$ ← wedge at base points.

<u>Ex</u>: $S^1 \wedge S^1 = S^2$

$S^n \wedge S^m = S^{n+m}$

Using K_k with $K_k^{(k-1)} = \{pt\}$, see $K_n \wedge K_m$ is $n+m-1$ conn.

<u>Claim</u>: $\pi_{n+m}(K_n \wedge K_m) = R \otimes_{\mathbb{Z}} R$.

<u>Pf</u>. Same as $\tilde{H}_{n+m}(K_n \wedge K_m; \mathbb{Z})$ by Hurewicz. By full Künneth thm for smash products (Hatcher pg 276), get $\tilde{H}_{n+m}(K_n \wedge K_m) = \bigoplus_{i=0}^{n+m} \tilde{H}_i(K_n) \otimes \tilde{H}_{n+m-i}(K_n)$
$= \tilde{H}_n(K_n) \otimes_{\mathbb{Z}} \tilde{H}_m(K_m)$ as the Tor term is 0. ▨

[Point: With reduced (co)homology, $X \wedge Y$ is more nat'l than $X \times Y$.]

Define $K_n \wedge K_m \xrightarrow[\mu]{} K_{n+m}$ so that μ_* on π_{n+m} is ring mult $R \otimes_{\mathbb{Z}} R \to R$.

[On HW will show $\phi: G \to H$ induces $K(G,n) \to K(H,n)$, this is the same idea.]

If $f: X \to K_n$ and $g: Y \to K_m$, then the cross prod $[f] \times [g]$ in $H^{n+m}(X \times Y)$ is the composition

$$X \times Y \xrightarrow[f \times g]{} K_n \times K_m \longrightarrow K_n \wedge K_m \xrightarrow[\mu]{} K_{n+m}$$

If $g: X \to K_m$, then $[f] \cup [g]$ is the composition

$$X \xrightarrow[\substack{\Delta \\ \text{diagonal}}]{} X \times X \xrightarrow[f \wedge g]{} K_n \wedge K_m \xrightarrow[\mu]{} K_{n+m}.$$

Can check the basic props of the cup prod this way. Pesky sign comes down to $S^m \wedge S^n \longrightarrow S^n \wedge S^m$ has degree $(-1)^{n \cdot m}$.

That this is really the cup product follows from confirming this for the α_n in $H^n(K_n; G)$.