More on fiber bundles.

Suppose $G$ is a topological group acting on a space $X$.

A fiber bundle with fiber $F$ and structure group $G$ is a map $p: E \to B$ together with a collection of homeomorphisms

$$\{ \varphi: p^{-1}(U) \to U \times F \}$$

for certain $U \subseteq B$ where:

1) The $U$ cover $B$.

2) Each $p^{-1}(U) \to U \times F$ commutes

$$\xymatrix{ p^{-1}(U) \ar[r]^p \ar[d]_{\text{proj}_U} & U \times F \ar[d]^{\text{proj}_U} \\

\text{proj}_U}

3) If $(U, \varphi)$ and $(V, \psi)$ are charts, then a continuous map $\Theta: U \cap V \to G$ so that $\forall x \in U \cap V$ and $f \in F$ have

$$(x, \Theta(x).f) = \varphi(\psi^{-1}(\varphi^{-1}(x,f)))$$

4) If $(U, \varphi)$ is a chart and $V \subseteq U$ open so is $(V, \varphi|_V)$.

5) The collection $(U, \varphi)$ is maximal with respect to (1-4).

6) $G$ acts effectively on $F$, i.e. $G \to \text{Homeo}(F)$.

Principal bundles: $G$ = any top gp, $F = G$ acted on by

left trans.

Construction: $\rho: \pi_1 B \to G$ $\tilde{E} = \tilde{B} \times G$ acted on by $\pi_1 B$ via $x \cdot (\tilde{b}, g) = (x \cdot \tilde{b}, \rho(x) \cdot g)$

$$E = \tilde{B} \times G$$

$$p: E \to B$$

$$[(\tilde{b}, g)] \mapsto \pi(\tilde{b})$$
This gives a flat principle bundle, since $\Theta$ is locally constant.

\[ U \times G \xleftarrow{\varphi_1^{-1}(U)} \xrightarrow{\varphi_2^{-1}} U \times G \overset{\text{cor to } U_1}{\text{cor to } U_2} \]

Then $\Theta : U \to G$ is the const $\rho(\text{g})$, where $\text{g} \in \pi_1 B$ sends $U_2 \to U_1$. Then $\varphi_1 \circ \varphi_2^{-1}(x, \text{g}) = (x, \Theta(x) \cdot \text{g})$.

Note: For a flat princ. bundle, if $U$ is simply conn then an idm of $\varphi^{-1}(x_0 \in U)$ with $G$ gives a unique trivialization of $\varphi^{-1}(U)$ as $U \times G$.

Cf: Flat connection.

Generalization: $G$ acts on $F$, $\rho : \pi_1 B \to G$.

Get a fiber bundle $E = \pi_1 B \setminus \widetilde{B} \times F$ where $\text{g} \in \pi_1 B$ acts by $(\tilde{b}, f) \mapsto (\text{g} \cdot \tilde{b}, \rho(\text{g}) \cdot f)$.

$E_x : B = S^1, G = \mathbb{Z}/2\mathbb{Z} = \{\pm 1\}, F = [-1, 1]$ $g \cdot f = gf$

Then $E(\rho, F) = \mathbb{R}$.
If $G$ is discrete, a principal $G$-bundle is a regular covering space corresponding to a homomorphism $\pi_1 B \to G$.

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Pull backs: $p: E \to B$ fiber bundle. Given $f: \mathbb{B} A \to B$, define $f^*(E) = \{(a, e) \mid f(a) = p(e)\}$. HW: The map $\Pi_A: f^*(E) \to A$ is a fiber bundle with the same fiber and structure group $E \to B$.

Universal Bundles: If $G$ is a topological group, there exists a principal $G$-bundle $EG \to BG$ so that for CW's $X$ we have

$$ [X, BG] \cong \text{Isomorphism classes of principal } G\text{-bundles over } B $$

$$(f: X \to BG) \mapsto f^*(EG)$$

If $G$ has the discrete topology, then $BG = K(G, 1)$ and $EG \to BG$ is as in the 1st proof of the existence of $K(G, 1)$'s.
Ex: \( G = \mathbb{Z} \) with the discrete topology.

\( BG = K(\mathbb{Z}, 1) = S^1 \)

Q: What is \( EG? \)  \( A: \mathbb{R} \)

For any \( X \), what map \( f: X \to S^1 \) gives the trivial bundle? It's the constant map.

Ex: \( G = S^1 \) with usual topo. Then \( BG = \mathbb{C}P^\infty \).

Recall \( S^1 \hookrightarrow S^{2n+1} \) by viewing \( S^{2n+1} \subseteq \mathbb{C}^{n+1} \) and \( S^1 \) acts diagonally.

Then have \( S^1 \hookrightarrow \mathbb{C}P^\infty \) as the universal \( S^1 \)-bundle \((= EG)\).

\( \downarrow \)

\[ \mathbb{C}P^\infty \]

Note: Cohomology of \( BG \) leads to characteristic classes of fiber bundles. E.g. any \( S^1 \)-fiber bundle has an Euler class \( c \in H^2(X; \mathbb{Z}) \) coming from the gen of \( H^2(\mathbb{C}P^\infty; \mathbb{Z}) \).

Next topic: Homology with local coeffs (§3.1 in Hatcher)

Bundle of gps: \( p: E \to B \) with fiber a discrete abelian gp \( G \) and str. gp \( \text{Aut}(G) \) with the discrete top.

Ex: \( M \) an \( n \)-mfld. \( R \) a ring. \( M_R = \{ \alpha \in H^n(M, M\setminus \{x\}; R) \mid x \in M \} \)

\[ \downarrow \]

\[ M \]

\[ \{x\} \]